Matrix orthogonal polynomials whose derivatives are also orthogonal

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Abstract

In this paper we prove some characterizations of the matrix orthogonal polynomials whose derivatives are also orthogonal, which generalize other known ones in the scalar case. In particular, we prove that the corresponding orthogonality matrix functional is characterized by a Pearson-type equation with two matrix polynomials of degree not greater than 2 and 1. The proofs are given for a general sequence of matrix orthogonal polynomials, not necessarily associated with an hermitian functional. However, we give several examples of non-diagonalizable positive definite weight matrices satisfying a Pearson-type equation, which show that the previous results are non-trivial even in the positive definite case.

A detailed analysis is made for the class of matrix functionals which satisfy a Pearson-type equation whose polynomial of degree not greater than 2 is scalar. We characterize the Pearson-type equations of this kind that yield a sequence of matrix orthogonal polynomials, and we prove that these matrix orthogonal polynomials satisfy a second order differential equation even in the non-hermitian case. Finally, we prove and improve a conjecture of Durán and Grünbaum concerning the triviality of this class in the positive definite case, while some examples show the non-triviality for hermitian functionals which are not positive definite.

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1 Introduction

The results published by Durán in [10] can be considered the starting point for a general study of matrix orthogonal polynomials satisfying differential equations. After [10], many other papers on the subject have appeared trying to find the similarities and main differences with respect to the classical and semi-classical scalar orthogonal polynomials (see [4, 5, 6, 7, 11, 12, 14, 16]). In spit of these efforts, a complete Bochner-type classification of matrix orthogonal polynomials satisfying second order differential equations similar to the scalar case (see [1, 2]) is far from being obtained.

However, the are many other differential properties that characterize the classical scalar orthogonal polynomials and that could lead to interesting matrix generalizations. These generalizations could clarify the structure of certain families of matrix orthogonal polynomials, being a source of properties for such families, as in the scalar case. Eventually, the understanding of these other differential properties could shed light on the structure of some families of matrix orthogonal polynomials satisfying differential equations, helping to find classification theorems.

It is well known that, apart from the second order differential equation, the classical scalar orthogonal polynomials (P_n) can be characterized by the orthogonality of their derivatives (P'_{n+1}) (see [3, 8, 17, 19, 20]) or, equivalently, by a linear relation between P_n and P'_{n+1}, P'_n, P'_{n-1} (see [18]). Also, these properties are equivalent to a Pearson-type equation for the corresponding orthogonality functional (see [8, 19, 20, 22]). The main objective of this paper is to prove that the equivalence between these three properties hold in the matrix case too (see Theorem 3.14).

The proofs of the above equivalences are given for any sequence of matrix orthogonal polynomials, not necessarily related to an hermitian weight matrix. Consequently, the Pearson-type equation must involve a distributional derivative. The distributional definition of the derivative not only permits to prove the results in a more general context, but unifies many different situations that otherwise would require a separate discussion. The reason is that the distributional Pearson-type equation takes care, not only of the first order differential equation for the weight, but of the necessary additional boundary conditions too (see Remark 2.9). So, the introduction of the distributional derivative becomes an advantage that permits to obtain more general results and, at the same time, in a simpler and more elegant way.

Diagonalizable matrix orthogonal polynomials (we will be more precise

about this concept later) are nothing really different from scalar orthogonal polynomials. So, the relevance of the results proved in this paper depends on the existence of non-diagonalizable examples of matrix orthogonal polynomials whose derivatives are also orthogonal. Examples 2, 3 and 4 show that there are non-diagonalizable positive definite weight matrices whose orthogonal polynomials enjoy such a property.

The weight matrix given in Example 2

$$e^{-x^2}\begin{pmatrix} 1+|a|^2x^2 & ax \\ \bar{a}x & 1 \end{pmatrix} dx, \quad x\in\mathbb{R}, \quad a\in\mathbb{C}\setminus\{0\},$$

appeared previously in [14] as an archetype of positive definite weight matrices whose orthogonal polynomials satisfy a second order differential equation. Curiously, the authors declare in [14], Section 7, Proposition 7.3, that the derivatives of these matrix orthogonal polynomials are no longer orthogonal with respect to any weight matrix, arguing that a contradiction appears when supposing a three term recurrence relation for such derivatives. However, if one makes the proposed computations in [14], Proposition 7.3, no contradiction appears! Indeed, we will see that this weight matrix satisfies a Pearson-type equation that, according to Theorem 3.14, implies the orthogonality of the derivatives of its orthogonal polynomials. Even more, we will find the positive definite weight matrix that gives the orthogonality of these derivatives.

The purpose of [14], Section 7, was to show that the equivalent characterizations of the classical scalar orthogonal polynomials do not necessarily hold for matrix orthogonal polynomials satisfying second order differential equations. It seems that the authors were not too lucky in the choice of the weight matrix since, if they had chosen the other example that they present, namely,

$$e^{-x^2}\begin{pmatrix} 1+|a|^2x^4 & ax^2 \\ \bar{a}x^2 & 1 \end{pmatrix} dx, \quad x\in\mathbb{R}, \quad a\in\mathbb{C}\setminus\{0\},$$

they would have succeeded. The reason is that, as can be easily checked, this other weight does not satisfy the required Pearson-type equation and, then, Theorem 3.14 implies that the derivatives of its orthogonal polynomials can not be orthogonal.

A particular class of the family of matrix orthogonal polynomials with orthogonal derivatives permits a deeper analysis. This is the class corresponding to a Pearson-type equation involving a scalar polynomial α under the derivative. These matrix orthogonal polynomials can be classified analogously to the classical scalar case, according to the roots of α: Hermite (no

roots), Laguerre (a simple root), Jacobi (two different roots) or Bessel-type (a double root). Moreover, a change of variable can reduce the different types to the canonical cases $\alpha(x) = 1, x, 1 - x^2, x^2$.

For this special class we develop explicit formulas for the related matrix parameters, such as the norm of the monic orthogonal polynomials, the coefficients of the three term recurrence relation or the coefficients of the linear relation between the polynomials and their derivatives. These formulas, although generalizations of the known ones in the classical scalar case, are more intricate due to the non-commutativity of the matrix product. However, they are very useful since they allow to characterize the Pearson-type equations that have a quasi-definite solution. In other words, if a matrix functional satisfies this kind of Pearson-type equation, we have a criterion to know if it generates a sequence of orthogonal polynomials (see Theorem 4.1). Notice that the importance of this result relies on the fact that we are dealing with general matrix functionals and not only with positive definite weight matrices, since the last ones always have an associated sequence of matrix orthogonal polynomials.

We also prove that the matrix orthogonal polynomials of the above class satisfy a second order differential equation with polynomial coefficients (see Theorems 4.3 and 4.4). The result is again true no matter if the corresponding orthogonality matrix functional is hermitian or not. This is one of the novelties of this result, since the previous works on differential equations for matrix orthogonal polynomials always dealt with the hermitian case only. Indeed, if we believe a conjecture formulated by Durán and Grünbaum in [13], this discovering is only relevant for the functionals of the referred class that are not positive definite. This conjecture says that every positive definite weight matrix in this class is diagonalizable. We present a proof of this conjecture (see Corollary 4.11).

The above conjecture was supported on a partial proof given in [13], that was incomplete due to the strong assumptions made there. First of all, it was supposed that the coefficients of the matrix polynomial appearing in the Pearson-type equation commute. Second, the proof was given separately for each of the canonical types of hermitian weight matrices that in the scalar case are positive definite: $\alpha(x) = 1, x, 1 - x^2$. So, the case $\alpha(x) = x^2$ is not considered, although the authors do not prove its incompatibility with a positive definite weight in the matrix case too. Finally, there is another less evident inconvenient. If α has a complex root, the required change of variable to arrive at a canonical situation destroys in general the hermiticity of the weight matrix. This means that, apart from the previous restrictions, the proof is only valid for the case of α with real roots. Our proof avoid all these

problems. Even more, we get a result that improves the one conjectured in [13] (see Theorem 4.10). In spite of this result, the non-triviality of the class under consideration is ensured by the existence of non-diagonalizable matrix orthogonal polynomials in such a class, even in the hermitian case (see [5, 13] and Example 5 of this paper).

The exposition of the above results will be structured in the following way along the paper. Section 2 introduces the notation, as well as some preliminary results and considerations that will of interest for the rest of the paper. In Section 3 we study the matrix orthogonal polynomials (P_n) with respect to a functional satisfying a Pearson-type equation with two matrix polynomials of degree not greater than 2 and 1. We prove that such a Pearson-type equation is equivalent to the orthogonality of the derivatives (P'_{n+1}) and, also, to a linear relation between P_n and P'_{n+1}, P'_n, P'_{n-1}. Some two-dimensional non-diagonalizable examples of positive definite weight matrices whose orthogonal polynomials satisfy these properties are presented at the end of the section. Section 4 is devoted to the analysis of the special case in which the polynomial under the derivative in the Pearson-type equation is a scalar one. We obtain the characterization of the Pearson-type equations of this kind with quasi-definite solutions, the differential equation for the related matrix orthogonal polynomials and the proof of the Durán-Grünbaum conjecture, finishing with some non-diagonalizable examples. Finally, in Section 5 we discuss the relation of the above results with other ones in the literature about second order differential equations for matrix orthogonal polynomials.

2 The Basics

We start with some notations and a summary of basic results that we will use in the rest of the paper.

In what follows, $\mathbb{C}^m$ will be the set of complex vectors of m components and $\mathbb{C}^{(m,m)}$ the set of $m \times m$ complex matrices. We shall denote by $\mathbb{P}^{(m)}$ the $\mathbb{C}^{(m,m)}$-left-module

$$\mathbb{P}^{(m)} = \left\{ \sum_{k=0}^{n} \alpha_k x^k \,\middle|\, \alpha_k \in \mathbb{C}^{(m,m)},\ n \in \mathbb{N} \right\},$$

and by means of $\mathbb{P}^{(m)'}$ the $\mathbb{C}^{(m,m)}$-right-module $\mathrm{Hom}\big(\mathbb{P}^{(m)}, \mathbb{C}^{(m,m)}\big)$. $\mathbb{P}_n^{(m)}$ will be the subset of matrix polynomials of $\mathbb{P}^{(m)}$ with degree not greater than n. In the scalar case $(m = 1)$ we will just write $\mathbb{P}^{(1)} = \mathbb{P}$ and $\mathbb{P}_n^{(1)} = \mathbb{P}_n$.

For all $P \in \mathbb{P}^{(m)}$ and $u \in \mathbb{P}^{(m)'}$ the duality bracket is defined by $\langle P, u \rangle = u(P)$ and it verifies the usual bilinear properties.

For $k \in \mathbb{N}$ and $u \in \mathbb{P}^{(m)'}$ the linear functional $ux^kI \in \mathbb{P}^{(m)'}$ is given by

$$\langle P, ux^kI \rangle = \langle x^kP, u \rangle,$$

where I denotes the $m \times m$ identity matrix. A linear extension gives the right-product $uQ \in \mathbb{P}^{(m)'}$ for $u \in \mathbb{P}^{(m)'}$, $Q \in \mathbb{P}^{(m)}$, with $Q(x) = \sum_{k=0}^{n} q_k x^k$, $q_k \in \mathbb{C}^{(m,m)}$, in the following way:

$$\langle P, uQ \rangle = \sum_{k=0}^{n} \langle x^kP, u \rangle q_k.$$

Similarly, the left-product $Qu \in \mathbb{P}^{(m)'}$ is defined by

$$\langle P, Qu \rangle = \langle PQ, u \rangle.$$

Every functional $u \in \mathbb{P}^{(m)'}$ induces a matrix inner product in $\mathbb{P}^{(m)}$ given by $\langle P, Q \rangle_u = \langle P, uQ^* \rangle$, where $Q^*(x) = \sum_{k=0}^{n} q_k^* x^k$ and q_k^* is the adjoint matrix of q_k. This matrix inner product enjoys the standard sesquilinear properties. The orthogonality with respect to u means the orthogonality with respect to this inner product.

The functional u^* is defined by

$$\langle P, u^*Q \rangle = \langle Q^*, uP^* \rangle^*,$$

and we will say that u is an hermitian functional if $u = u^*$. In this case $\langle P, uP^* \rangle$ is hermitian for any $P \in \mathbb{P}^{(m)}$. We will say that an hermitian functional u is positive definite if $\langle P, uP^* \rangle$ is positive definite for every $P \in \mathbb{P}^{(m)}$ with $\det P \neq 0$. In what follows we denote this condition by $u > 0$. In the same way, for a positive definite matrix A we will write $A > 0$.

We denote by $\mu_k = \langle x^kI, u \rangle$ the k-th moment with respect to $u \in \mathbb{P}^{(m)'}$. Given a sequence $(\mu_k)_{k \geq 0}$ in $\mathbb{C}^{(m,m)}$, there exists a unique $u \in \mathbb{P}^{(m)'}$ such that $\langle x^kI, u \rangle = \mu_k$.

If $u \in \mathbb{P}^{(m)'}$ has moments $(\mu_k)_{k \geq 0}$, we say that u is quasi-definite (or non-singular) if $\det \Delta_n \neq 0$ for $n \geq 0$, where Δ_n is the Hankel-block matrix

$$\Delta_n = \begin{pmatrix} \mu_0 & \mu_1 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_{n+1} \\ \cdots & \cdots & \cdots & \cdots \\ \mu_n & \mu_{n+1} & \cdots & \mu_{2n} \end{pmatrix}.$$

Notice that u is hermitian if and only if $\mu_n = \mu_n^*$ for $n \geq 0$, or, equivalently, $\Delta_n = \Delta_n^*$ for $n \geq 0$.

The interest of the quasi-definite functionals relies on the following result (see [9, 15, 21]).

Theorem 2.1. *$u \in \mathbb{P}^{(m)'}$ is quasi-definite if and only if there exists a sequence $(P_n)_{n\geq 0}$ of left orthogonal matrix polynomials with respect to u, that is:*

(i) $P_n \in \mathbb{P}^{(m)}$, $\deg P_n = n$.

(ii) The leading coefficient of P_n is non-singular.

(iii) $\langle x^k P_n, u\rangle = E_n \delta_{nk}$, with E_n non-singular, for $0 \leq k \leq n$.

Moreover, the sequence $(P_n)_{n\geq 0}$ is unique up to non-singular left matrix factors and verifies a recurrence relation

$$xP_n(x) = \alpha_n P_{n+1}(x) + \beta_n P_n(x) + \gamma_n P_{n-1}(x),$$

where $P_0 \in \mathbb{C}^{(m,m)}$ is non-singular, $P_{-1} = 0$ and α_n, β_n, $\gamma_n \in \mathbb{C}^{(m,m)}$, with α_n, γ_n non-singular.

The last result of this theorem has a converse (Favard's Theorem): for any sequence $(P_n)_{n\geq 0}$ verifying the above recurrence relation there exists a unique (up to non-singular right matrix factors) quasi-definite functional u such that $(P_n)_{n\geq 0}$ is its sequence of left orthogonal matrix polynomials (see [9, 15, 21]). Analogously we can define the right orthogonal matrix polynomials with respect to u, which are the adjoints of the left orthogonal polynomials associated with u^*. In what follows we will consider only left orthogonal matrix polynomials, and we will call them just matrix orthogonal polynomials (MOP).

Remark 2.2. Given a functional $u \in \mathbb{P}^{(m)'}$, we can normalize the corresponding MOP by choosing the only monic ones $(P_n)_{n\geq 0}$. In what follows we will assume this choice, so, a unique sequence of non-singular matrices $(E_n)_{n\geq 0}$, $E_n = \langle x^n P_n, u\rangle$, is associated with any quasi-definite functional u. Also, β_n and γ_n will denote the matrix coefficients of the related recurrence relation

$$xP_n(x) = P_{n+1}(x) + \beta_n P_n(x) + \gamma_n P_{n-1}(x).$$

Similarly, given a sequence MOP, we can normalize the corresponding functional u in different ways, for instance, by requiring $\langle I, u\rangle = I$. However, we will not fix the normalization for the moment because the most convenient one depends on the problem that we wont to study.

In the case of non quasi-definite functionals, the full sequence of MOP does not exist. Nevertheless, we have the following general result.

Proposition 2.3. *For every $u \in \mathbb{P}^{(m)'}$ the following statements are equivalent:*

(i) $\Delta_0, \dots, \Delta_n$ *are non-singular.*

(ii) There exists a finite segment $(P_k)_{k=0}^n$ *of monic MOP with respect to* u*, that is:*

(a) $P_k \in \mathbb{P}^{(m)},\ \deg P_k = k$.

(b) $\langle x^j P_k, u\rangle = E_k \delta_{kj}$*, with* E_k *is non-singular, for* $0 \le j \le k \le n$.

Moreover, under the above conditions, the segment $(P_k)_{k=0}^n$ *is unique and there exists a unique monic polynomial* P_{n+1} *whit* $\deg P_{n+1} = n+1$ *such that* $\langle x^j P_{n+1}, u\rangle = 0$ *for* $0 \le j \le n$.

Proof. Suppose that $\Delta_0, \dots, \Delta_n$ are non-singular. If $P_k(x) = \sum_{i=0}^k \pi_i^{(k)} x^i$, $\pi_i^{(k)} \in \mathbb{C}^{(m,m)}$, then, $\langle x^j P_k, u\rangle = \sum_{i=0}^k \pi_i^{(k)} \mu_{i+j}$. Choosing $\pi_k^{(k)} = I$, the system $\sum_{i=0}^k \pi_i^{(k)} \mu_{i+j} = 0,\ j = 0, \dots, k-1$, can be represented as

$$\left(\pi_0^{(k)},\, \pi_1^{(k)},\, \dots,\, \pi_{k-1}^{(k)} \right) \Delta_{k-1} = -\left(\mu_k,\, \mu_{k+1},\, \dots,\, \mu_{2k-1} \right),$$

which has a unique solution for $k = 0, 1, \dots, n+1$.

On the other hand, E_k is non singular for $k = 0, 1, \dots, n$. In fact, we have $\langle x^j P_k, u\rangle = E_k \delta_{kj},\ j = 0, \dots, k,\ k = 0, \dots, n$, and, so,

$$\left(\pi_0^{(k)},\, \pi_1^{(k)},\, \dots,\, \pi_{k-1}^{(k)},\, I \right) \Delta_k = \left(0,\, 0,\, \dots,\, 0,\, E_k \right).$$

If E_k is singular, there exists $v \in \mathbb{C}^m \setminus \{0\}$ such that $v^T E_k = 0$. Hence,

$$\left(v^T \pi_0^{(k)},\, v^T \pi_1^{(k)},\, \dots,\, v^T \pi_{k-1}^{(k)},\, v^T \right) \Delta_k = \left(0,\, 0,\, \dots,\, 0,\, 0 \right),$$

and this result contradicts the non-singularity of Δ_k for $k = 0, \dots, n$.

For the converse, let us suppose that there exists a finite segment $(P_k)_{k=0}^n$ of MOP with respect to u with $E_k = \langle x^k P_k, u\rangle$. It is easy to see that the conditions $\langle x^j Q_k, u\rangle = E_k \delta_{kj},\ j = 0, \dots k$, where $Q_k \in \mathbb{P}_k^{(m)}$, ensures that $Q_k = P_k,\ \ k = 0, \dots, n$. Writing $Q_k(x) = \sum_{i=0}^k \pi_i^{(k)} x^i$, the above assertion means that, for $k = 0, \dots, n$, the system

$$\left(\pi_0^{(k)},\, \pi_1^{(k)},\, \dots,\, \pi_{k-1}^{(k)},\, \pi_k^{(k)} \right) \Delta_k = \left(0,\, 0,\, \dots,\, 0,\, E_k \right)$$

has a unique solution and, hence, Δ_k is non-singular. $\square$

Concerning the partial hermiticity of a functional, we have the following immediate result.

Proposition 2.4. *Let $u \in \mathbb{P}^{(m)'}$. If $(p_k)_{k=0}^n$ is a basis of $\mathbb{P}_n^{(m)}$, $\Delta_n = \Delta_n^*$ if and only if $(\langle p_k, up_j^* \rangle)_{k,j=0}^n$ is hermitian.*

In particular, if u has a finite segment $(P_k)_{k=0}^n$ of MOP,

$$\Delta_n = \Delta_n^* \iff \langle P_k, uP_j^* \rangle = E_k \delta_{kj}, \quad E_k = E_k^*, \quad 0 \leq j, k \leq n.$$

The second assertion of the above proposition says that, when $\Delta_0, \ldots, \Delta_n$ are non-singular, the condition $\Delta_n = \Delta_n^*$ means that the finite segments of left and right orthogonal matrix polynomials are each one the hermitian adjoint of the other one.

Also, for the hermitian positive definite functionals on $\mathbb{P}_n^{(m)}$ we have the following characterization.

Proposition 2.5. *Let $u \in \mathbb{P}^{(m)'}$. If $(p_k)_{k=0}^n$ is a basis of $\mathbb{P}_n^{(m)}$, the following statements are equivalent:*

(i) $\Delta_n > 0$.

(ii) $(\langle p_k, up_j^* \rangle)_{k,j=0}^n > 0$.

(iii) u has a finite segment $(P_k)_{k=0}^n$ of MOP such that $\langle P_k, uP_j^* \rangle = E_k \delta_{kj}$ with $E_k > 0$ for $0 \leq j, k \leq n$.

(iv) $\langle P, uP^* \rangle > 0$ for any $P \in \mathbb{P}_n^{(m)}$ such that $\det P \neq 0$.

Proof. We only prove *(i)* $\Leftrightarrow$ *(iv)*, since the rest of equivalences are immediate. For any matrix polynomial $P(x) = \sum_{i=0}^k A_i x^i$, $A_i \in \mathbb{C}^{(m,m)}$, $k \leq n$,

$$\langle P, uP^* \rangle = (\, A_1 \; \ldots \; A_k \,)\, \Delta_k \begin{pmatrix} A_1^* \\ \vdots \\ A_k^* \end{pmatrix}.$$

So, $\langle P, uP^* \rangle$ is hermitian if Δ_n is hermitian. If $v \in \mathbb{C}^m$,

$$v^* \langle P, uP^* \rangle v = (\, v_0^* \; \ldots \; v_k^* \,)\, \Delta_k \begin{pmatrix} v_0 \\ \vdots \\ v_k \end{pmatrix}, \qquad v_i = A_i^* v. \tag{1}$$

Then, if $v \neq 0$, $\det P \neq 0$ implies $v_i \neq 0$ for some i. So, equality (1) gives $v^* \langle P, uP^* \rangle v > 0$ if $\Delta_n > 0$.

For the converse, if $\langle P, uP^* \rangle$ is hermitian for $P \in \mathbb{P}_n^{(m)}$ with $\det P \neq 0$, $\mu_{2k} = \langle x^k I, ux^k I \rangle = \mu_{2k}^*$ for $k \leq n$. Besides, $\mu_{2k-1} = \mu_{2k-1}^*$ for $k \leq n$ too,

due to the identity $\langle (x^k + x^{k-1})I, u(x^k + x^{k-1})I\rangle = \mu_{2k} + \mu_{2k-2} + 2\mu_{2k-1}$. Therefore $\Delta_n = \Delta_n^*$.

Suppose $\langle P, uP^*\rangle > 0$ for any $P \in \mathbb{P}_n^{(m)}$ with $\det P \neq 0$. Let $(\, v_0 \dots v_k \,)$, $v_i \in \mathbb{C}^m$, with $v_k \neq 0$ and $k \leq n$. We can always find $A_i \in \mathbb{C}^{(m,m)}$ such that $A_i^* v_k = v_i$, $A_k = I$. The polynomial $P(x) = \sum_{i=0}^k A_i x^i$ lies on $\mathbb{P}_n^{(m)}$ and $\det P \neq 0$. So, relation (1) gives

$$(\, v_0^* \, \dots \, v_k^* \,)\, \Delta_k \begin{pmatrix} v_0 \\ \vdots \\ v_k \end{pmatrix} > 0, \quad \text{if } v_k \neq 0, \quad k \leq n.$$

This proves by induction that $\Delta_n > 0$. $\square$

Remark 2.6. Notice that, if u is an hermitian and positive definite functional, then it is quasi-definite. So, there exits the corresponding sequence $(P_n)_{n\geq 0}$ of MOP with E_n hermitian and positive definite.

Similarly to the scalar case, the positive definite matrix functionals are those ones given by

$$\langle P, u\rangle = \int P(x)\, dM(x), \tag{2}$$

where dM is a positive definite weight matrix on $\mathbb{R}$, that is, a positive definite matrix of measures supported on the real line ($M(S)$ is positive semidefinite for any Borel set $S \subset \mathbb{R}$) with finite moments $\int x^n dM(x)$, $n \geq 0$, and such that $\int P(x)\, dM(x)\, P(x)^*$ is non-singular if $\det P \neq 0$ (see [9]). This is, for instance, the case of an absolutely continuous matrix of measures $dM(x) = W(x)\, dx$ with finite moments, $W(x)$ being semidefinite positive for any $x \in \mathbb{R}$ and non-singular for infinitely many points of the real line.

In what follows we will identify any $m \times m$ matrix dM of measures on $\mathbb{C}$ with finite moments (not necessarily hermitian), and the functional $u \in \mathbb{P}^{(m)'}$ defined by (2). Thus, we will write $u = dM$ for such a functional.

A specially interesting family of matrix functionals is given by the functionals which satisfy a differential equation of Pearson-type (see [4, 5]). The definition of this family requires the introduction of the derivative operator in the space $\mathbb{P}^{(m)'}$, which is the linear operator $D\colon \mathbb{P}^{(m)'} \to \mathbb{P}^{(m)'}$ such that

$$\langle P, Du\rangle = -\langle P', u\rangle.$$

The equality $D(u\Phi) = (Du)\Phi + u\Phi'$ holds for all $u \in \mathbb{P}^{(m)'}$ and $\Phi \in \mathbb{P}^{(m)}$.

Definition 2.7. Let $u \in \mathbb{P}^{(m)'}$. We say that $u \in \mathcal{P}$ or, equivalently, u is a $\mathcal{P}$-functional, if there exist $\Phi, \Psi \in \mathbb{P}^{(m)}$, with $\det \Phi \neq 0$, such that

$$D\,(u\Phi) = u\Psi \qquad \text{(Pearson-type equation)}$$

If $\deg \Phi \leq p$ and $\deg \Psi \leq q$, we say that $u \in \mathcal{P}_{p,q}$ or u is a $\mathcal{P}_{p,q}$-functional. In both cases we also say that the corresponding sequence of MOP belongs to the family $\mathcal{P}$ or $\mathcal{P}_{p,q}$ respectively.

Remark 2.8. The condition $\det \Phi \neq 0$ is imposed to avoid any triviality of the definition, ensuring that it involves all the components $u_{ij}\colon \mathbb{P}^{(m)} \to \mathbb{C}$ of $u = (u_{ij})_{i,j=0}^{m}$. Notice that

$$\det \Phi = 0 \iff \Phi v = 0 \text{ for some } v \in \mathbb{C}^m[x] \setminus \{0\}.$$

In fact, if $\Phi v = 0$ for some $v \in \mathbb{C}^m[x] \setminus \{0\}$, then $0 = (\operatorname{adj} \Phi)\Phi v = (\det \Phi)v$. To see the converse, remember that every $\Phi \in \mathbb{P}^{(m)}$ can be factorized as $\Phi = P\hat{\Phi}Q$, with $\hat{\Phi} \in \mathbb{P}^{(m)}$ diagonal and $P, Q \in \mathbb{P}^{(m)}$ invertible, that is, $\det P, \det Q \in \mathbb{C} \setminus \{0\}$. Therefore, $\det \Phi = 0$ implies $\det \hat{\Phi} = 0$ and, since $\hat{\Phi}$ is diagonal, $\hat{\Phi} v_0 = 0$ for some $v_0 \in \mathbb{C}^m \setminus \{0\}$, which gives $\Phi v = 0$ with $v = Q^{-1} v_0 \in \mathbb{C}^m[x] \setminus \{0\}$.

Remark 2.9. The distributional definition of the derivative operator D implies that, in general, the Pearson-type equation involves, not only a relation between standard derivatives, but a boundary condition too. Consider, for instance, a functional $u = W(x)\,dx$, $x \in \Gamma$, with W an analytic matrix function on a regular curve Γ of the complex plane. Then, $Du = W'(x)\,dx + W(x)(\delta(x-a) - \delta(x-b))\,dx$, where a and b are the initial and end points of Γ respectively. So, if the curve is open, together with the equality $(W\Phi)' = W\Psi$, we need the boundary condition $(W\Phi)(a) = (W\Phi)(b) = 0$ to ensure the Pearson-type equation $D(u\Phi) = u\Psi$. The case of a closed curve does not need an additional boundary condition since we suppose that W is analytic on Γ. Moreover, in this case, the Pearson-type equation holds even if $(W\Phi)' \neq W\Psi$ but $(W\Phi)' - W\Psi$ is analytic on the region enclosed by Γ, due to Cauchy's Theorem. The Pearson-type equation can be satisfied if W is only analytic on $\Gamma \setminus \{a, b\}$ but the limits $(W\Phi)(a^+) := \lim_{t \to t_0} (W\Phi)(\gamma(t))$, $(W\Phi)(b^-) := \lim_{t \to t_1} (W\Phi)(\gamma(t))$ exist, where $\gamma\colon [t_0, t_1] \to \Gamma$ is a parametrization of Γ, $a = \gamma(t_0)$, $b = \gamma(t_1)$. Then,

$$D(u\Phi) = (W\Phi)'(x)\,dx + (W\Phi)(a^+)\,\delta(x-a)\,dx - (W\Phi)(b^-)\,\delta(x-b)\,dx,$$

so, we get the Pearson-type equation adding to $(W\Phi)' = W\Psi$ the boundary conditions

$$\begin{aligned} &(W\Phi)(a^+) = (W\Phi)(b^-) && \text{closed curve,} \\ &(W\Phi)(a^+) = (W\Phi)(b^-) = 0 && \text{open curve.} \end{aligned}$$

The distributional derivative not only unifies all these cases, but allows to consider more general situations, such as functionals defined by matrix measures supported on an arbitrary subset of the complex plane.

If $u \in \mathbb{P}^{(m)'}$ is a $\mathcal{P}$-functional with a Pearson-type equation $D\,(u\Phi) = u\Psi$, then, for every $\Omega \in \mathbb{P}^{(m)}$,

$$D\,(u\Phi\Omega) = u\left(\Phi\Omega' + \Psi\Omega\right). \tag{3}$$

Therefore, the set

$$\mathcal{M}(u) = \{\Phi \in \mathbb{P}^{(m)} \mid D(u\Phi) = u\Psi,\ \Psi \in \mathbb{P}^{(m)}\}$$

is a right-ideal of $\mathbb{P}^{(m)}$, but it is not necessarily principal, because the euclidean division algorithm is not valid in $\mathbb{P}^{(m)}$. This is an obstacle to find a canonical representative of $\mathcal{M}(u)$ that could lead to a classification of $\mathcal{P}$-functionals similarly to the scalar case.

Notice that $\mathcal{P} = \bigcup_{p,q\geq 0} \mathcal{P}_{p,q}$, and $\mathcal{P}_{p,q} \subset \mathcal{P}_{p',q'}$ if $p \leq p'$ and $q \leq q'$. The set

$$\mathcal{M}_{p,q}(u) = \{\Phi \in \mathbb{P}_p^{(m)} \mid D(u\Phi) = u\Psi,\ \Psi \in \mathbb{P}_q^{(m)}\}$$

is not an ideal of $\mathbb{P}^{(m)}$, but a $\mathbb{C}^{(m,m)}$-right-submodule of $\mathbb{P}_p^{(m)}$. Although it is finitely generated, it is not cyclic in general, what means again a problem for finding a canonical representative of $\mathcal{M}_{p,q}(u)$.

Example 1. Let us consider $u \in \mathbb{P}^{(2)'}$ given by

$$u = (1-x^2)\begin{pmatrix} 1+3x^2 & 2x \\ 2x & 1 \end{pmatrix} dx, \quad x \in (-1,1).$$

A direct computation shows that u is a $\mathcal{P}_{3,2}$-functional with

$$\mathcal{M}_{3,2}(u) = \operatorname{span}_{\mathbb{C}^{(2,2)}}\left\{(1-x^2)I, x(1-x^2)\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right\}$$

generated by two elements. Indeed, if

$$\Phi(x) = (1-x^2)\Lambda_1 + x(1-x^2)\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\Lambda_2, \quad \Lambda_i \in \mathbb{C}^{(2,2)},$$

then $D(u\Phi) = u\Psi$ with

$$\Psi(x) = \begin{pmatrix} -2x & 2 \\ 2-6x^2 & -8x \end{pmatrix} \Lambda_1 + \begin{pmatrix} 0 & 2x \\ 0 & 1-9x^2 \end{pmatrix} \Lambda_2.$$

We can get cyclic modules for u by going down in the net $(\mathcal{P}_{p,q})_{p,q\geq 0}$, but there are two different ways to do it. From the previous result we obtain

- $u \in \mathcal{P}_{2,2}$ with $\mathcal{M}_{2,2}(u) = \operatorname{span}_{\mathbb{C}^{(2,2)}} \left\{ (1-x^2)I \right\}$.
- $u \in \mathcal{P}_{3,1}$ with $\mathcal{M}_{3,1}(u) = \operatorname{span}_{\mathbb{C}^{(2,2)}} \left\{ (1-x^2) \begin{pmatrix} 3 & 0 \\ -2x & 1 \end{pmatrix} \right\}$.

In fact,

$$\begin{aligned} D\left(u(1-x^2)I\right) &= u \begin{pmatrix} -2x & 2 \\ 2-6x^2 & -8x \end{pmatrix}, \\ D\left(u(1-x^2)\begin{pmatrix} 3 & 0 \\ -2x & 1 \end{pmatrix}\right) &= u \begin{pmatrix} -10x & 2 \\ 4 & -8x \end{pmatrix}. \end{aligned}$$

This splitting shows clearly the problem of classification of $\mathcal{P}$-functionals. Moreover, we can not go down more than this in the net $(\mathcal{P}_{p,q})_{p,q\geq 0}$ since

$$\begin{aligned} &\mathcal{M}_{2,1}(u) = \mathcal{M}_{2,2}(u) \cap \mathcal{M}_{3,1}(u) = \operatorname{span}_{\mathbb{C}^{(2,2)}} \left\{ (1-x^2) \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}, \\ &\mathcal{M}_{1,2}(u) = \mathcal{M}_{3,0}(u) = \mathcal{M}_{0,3}(u) = \{0\}, \end{aligned}$$

and, hence, $u \notin \mathcal{P}_{p,q}$ for $p+q \leq 3$.

Notice that the above problems of classification happen even for quasi-definite functionals since our example was positive definite. However, if we restrict our attention to quasi-definite functionals, there is a singular situation. As we will prove later (see Theorem 3.4), if $\Delta_0, \Delta_1, \Delta_2$ are non-singular for some $u \in \mathcal{P}_{2,1}$, then $\mathcal{M}_{2,1}(u)$ is cyclic. This implies that we can associate with each sequence of MOP in the family $\mathcal{P}_{2,1}$ a canonical representative: the unique (up to non-singular right matrix factors) generator of $\mathcal{M}_{2,1}(u)$, u being the related orthogonality matrix functional.

A way to solve the problem of classification of $\mathcal{P}$-functionals uses the fact that $\mathcal{M}(u)$ always has a non-trivial scalar representative. In fact, choosing $\Omega = \operatorname{adj}\Phi$ in (3) gives $\Phi\Omega = (\det\Phi)I$, which yields the following characterization (see [4, 5]).

Proposition 2.10. *The functional $u \in \mathbb{P}^{(m)'}$ belongs to the family $\mathcal{P}$ if and only if there exist $\alpha \in \mathbb{P} \setminus \{0\}$ and $\Psi \in \mathbb{P}^{(m)}$ such that*

$$D(u\alpha I) = u\Psi.$$

Notice that the set

$$\widetilde{\mathcal{M}}(u) = \{\alpha \in \mathbb{P} \mid D(u\alpha I) = u\Psi,\ \Psi \in \mathbb{P}^{(m)}\}$$

is a non-trivial bilateral ideal of $\mathbb{P}$, which is, therefore, principal. So, there exists an $\alpha \in \mathbb{P} \setminus \{0\}$, unique up non-trivial factors in $\mathbb{C}$, that is generator of $\widetilde{\mathcal{M}}(u)$. This scalar generator can be used to classify the $\mathcal{P}$-functionals.

Definition 2.11. Let $u \in \mathbb{P}^{(m)'}$ be a $\mathcal{P}$-functional and let $\alpha \in \mathbb{P} \setminus \{0\}$ be a generator of $\widetilde{\mathcal{M}}(u)$. The class of u is $s = \max\{\deg \alpha - 2, \deg \Psi - 1\}$, where $\Psi \in \mathbb{P}^{(m)}$ is such that $D(u\alpha I) = u\Psi$.

The interesting $\mathcal{P}$-functionals are those ones that have a sequence of MOP, that is, the quasi-definite $\mathcal{P}$-functionals. These are called semi-classical functionals (see [4, 5]). As in the scalar case, the semi-classical functionals can be characterized by several differential properties of the corresponding MOP.

Theorem 2.12. *Let $u \in \mathbb{P}^{(m)'}$ be quasi-definite and let $(P_n)_{n\geq 0}$ be the associated sequence of MOP. Then, the following statements are equivalent:*

(i) $u \in \mathcal{P}$.

(ii) There exist $\alpha \in \mathbb{P} \setminus \{0\}$ and $\Theta_j^{(n)} \in \mathbb{C}^{(m,m)}$ such that

$$\alpha(x)P'_{n+1}(x) = \sum_{j=-s}^{\deg \alpha} \Theta_j^{(n)} P_{n+j}(x) \qquad \textit{(structure relation)}$$

with $s \geq \max\{\deg \alpha - 2, 0\}$ independent of n and $\Theta_{-s}^{(n)} \neq 0$ for some $n \geq s$.

(iii) There exist $a \in \mathbb{P} \setminus \{0\}$, $b \in \mathbb{P}$ and $\Lambda_k^{(n)} \in \mathbb{C}^{(m,m)}$ such that

$$a(x)P''_n(x) + b(x)P'_n(x) = \sum_{k=-r}^{r} \Lambda_k^{(n)} P_{n+k}(x) \qquad \textit{(differo-differential equation)}$$

with $r \geq \max\{\deg a - 2, \deg b - 1\}$ independent of n.

We use the convention $P_k = 0$ for $k < 0$.

Proof. See [4, 5]. □

Remark 2.13. Let us suppose that a $\mathcal{P}$-functional $u \in \mathbb{P}^{(m)'}$ satisfies a Pearson-type equation $D(u\alpha I) = u\Psi$, $\alpha \in \mathbb{P} \setminus \{0\}$, $\Psi \in \mathbb{P}^{(m)}$, and let $s = \max\{\deg \alpha - 2, \deg \Psi - 1\}$. Then, the proofs given in [5] show that the structure relation appearing in Theorem 2.12 *(ii)* is satisfied for the same polynomial α and integer s. However, contrary to the scalar case, the differo-differential equation given in Theorem 2.12 *(iii)* can not be ensured for $a = \alpha$, $r = s$, but for $a = \alpha^2$ and $r = \max\{2\deg\alpha - 2, 2s + 2\} = \max\{2\deg\alpha - 2, 2\deg\Psi\} \geq s$.

In the scalar case, the classical orthogonal polynomials can be characterized by a Pearson-type equation $D(u\alpha) = u\beta$, $\alpha \in \mathbb{P}_2 \setminus \{0\}$, $\beta \in \mathbb{P}_1$, for the corresponding orthogonality functional u. When trying to generalize the concept of classical orthogonal polynomials to the matrix case using a Pearson-type equation, the following two possibilities appear:

- Zero class: $u \in \mathbb{P}^{(m)'}$ belongs to the zero class if it is semi-classical with class $s = 0$, that is, u is quasi-definite and there exist $\alpha \in \mathbb{P}_2 \setminus \{0\}$, $\Psi \in \mathbb{P}_1^{(m)}$, such that $D(u\alpha I) = u\Psi$.

- Family $\mathcal{P}_{2,1}$: $u \in \mathbb{P}^{(m)'}$ is a $\mathcal{P}_{2,1}$-functional, or belongs to the family $\mathcal{P}_{2,1}$, if there exist $\Phi \in \mathbb{P}_2^{(m)}$, $\Psi \in \mathbb{P}_1^{(m)}$, with $\det\Phi \neq 0$, such that $D(u\Phi) = u\Psi$.

The MOP associated with zero class functionals or quasi-definite $\mathcal{P}_{2,1}$-functionals can be considered as matrix generalizations of the classical scalar orthogonal polynomials. Notice that a quasi-definite $\mathcal{P}_{2,1}$-functional is always semi-classical, but its class can be greater than zero. In fact, excepting the scalar case, the family of quasi-definite $\mathcal{P}_{2,1}$-functionals is strictly greater than the zero class, as can be seen in Examples 2, 3 and 4. Both, the family $\mathcal{P}_{2,1}$ and the zero class, are interesting sets of matrix functionals since the related MOP inherit some of the properties that characterize the classical orthogonal polynomials in the scalar case. This will be shown in the following sections, which are devoted to the study of the family $\mathcal{P}_{2,1}$ and the zero class.

Before doing that, we will comment some other questions of importance for matrix orthogonal polynomials. As we have pointed out, a central concept for matrix functionals is the diagonalizability or, more generally, the reducibility. We say that a functional $u \in \mathbb{P}^{(m)'}$ is diagonal or block-diagonal if its moment sequence $(\mu_n)_{n\geq 0}$ enjoys such a property. We write

$u = u^{(1)} \oplus \cdots \oplus u^{(k)}$ if $\mu_n = \mu_n^{(1)} \oplus \cdots \oplus \mu_n^{(k)}$, where $(\mu_n^{(i)})_{n \geq 0}$ are the moments of $u^{(i)}$.

To simplify the analysis of a matrix functional $u \in \mathbb{P}^{(m)'}$, the usual strategy is to connect it with a diagonal or block-diagonal one $\hat{u} \in \mathbb{P}^{(m)'}$ through a relation that permits to translate the information from $\hat{u}$ to u. For instance, if $\hat{u} = TuS$, with $T, S \in \mathbb{C}^{(m,m)}$ non-singular, we say that u is equivalent to $\hat{u}$. In particular, when $S = T^*$ we say that u is congruent to $\hat{u}$, while if $S = T^* = T^{-1}$ way say that u is unitarily similar to $\hat{u}$. Notice the difference with the terminology used by other authors, we prefer to preserve the usual one in Linear Algebra to avoid unnecessary confusion. A matrix functional is diagonalizable or reducible by equivalence if it is equivalent to a diagonal or block-diagonal one respectively. We define in a similar way the diagonalizability or reducibility by congruence and the unitary diagonalizability or reducibility.

A change of variable $t(x) = ax + b$, $a \in \mathbb{C} \setminus \{0\}$, $b \in \mathbb{C}$, can be used to relate matrix functionals too. Given $u \in \mathbb{P}^{(m)'}$ we define $u_t \in \mathbb{P}^{(m)'}$ by

$$\langle P, u_t \rangle = \langle P \circ t, u \rangle,$$

so that, if $u = dM$, then $u_t = d(M \circ t^{-1})$. Notice that, with this definition, $(Du)_t = (Du_t)t'$.

The kind of relation that we use depends on the properties that we need to preserve. For example, the equivalence transformation and the change of variable keep invariant the quasi-definite character, any family $\mathcal{P}_{p,q}$ as well as the class of a $\mathcal{P}$-functional (in fact, the MOP and the corresponding Pearson-type equations are trivially related by these transformations). This means that, concerning these properties, the only non-trivial matrix functionals are those ones that are not reducible by equivalence or change of variable. In particular, if we are going to study a characteristic of a functional u that only depends on such properties, then we can always use the normalization $\langle I, u \rangle = I$ since we can work, for example, with the equivalent functional $\hat{u} = u\mu_0^{-1}$. Also, this allows when studying zero class functionals to restrict our attention to the canonical choices $\alpha(x) = 1, x, 1 - x^2, x^2$ of the scalar polynomial in the Pearson-type equation, due to the freedom in the change of variables.

However, if we are interested in a characteristic that depends on the hermiticity or positive definiteness of u (or, more generally, on the hermiticity or positive definiteness of some moments μ_n or Hankel matrices Δ_n) we must use congruence transformations and changes of variable with real coefficients. This is the reason to avoid using the canonical forms of the scalar

polynomial α when studying hermitian zero class functionals, unless we are sure that α has real roots. Also, the normalization $\langle I, u\rangle = I$ can be used, while preserving any hermiticity property of u, whenever $\mu_0 > 0$ since, then, we can use the congruent functional $\hat{u} = L^{-1}u(L^{-1})^*$, where $\mu_0 = LL^*$ is the Cholesky factorization of μ_0.

3 The family $\mathcal{P}_{2,1}$

The aim of this section is to study the differential properties of the MOP associated with $\mathcal{P}_{2,1}$-functionals. The main result is Theorem 3.14, which shows that some characterizations of the classical scalar orthogonal polynomials remain true for the matrix family $\mathcal{P}_{2,1}$. Along the way to prove Theorem 3.14 we will obtain a chain of results which have their own interest.

We will start fixing some notations that we will need in the rest of the section. Let $u \in \mathbb{P}^{(m)'}$ be a $\mathcal{P}_{2,1}$-functional, that is, $D(u\Phi) = u\Psi$, where $\Phi(x) = \varphi_0 + \varphi_1 x + \varphi_2 x^2$, $\Psi(x) = \psi_0 + \psi_1 x$, with $\varphi_i, \psi_j \in \mathbb{C}^{(m,m)}$ and $\det\Phi \neq 0$. The above Pearson-type equation is equivalent to

$$n(\mu_{n-1}\varphi_0 + \mu_n\varphi_1 + \mu_{n+1}\varphi_2) = -(\mu_n\psi_0 + \mu_{n+1}\psi_1), \qquad n \geq 0, \tag{4}$$

where $(\mu_k)_{k\geq 0}$ are the moments of u and $\mu_{-1} = 0$. We denote

$$\widetilde{u} = u\Phi, \qquad \widetilde{\mu}_n = \langle x^n I, \widetilde{u}\rangle, \qquad \widetilde{\Delta}_n = \begin{pmatrix} \widetilde{\mu}_0 & \widetilde{\mu}_1 & \dots & \widetilde{\mu}_n \\ \dots & \dots & \dots & \dots \\ \widetilde{\mu}_n & \widetilde{\mu}_{n+1} & & \widetilde{\mu}_{2n} \end{pmatrix}.$$

The moments of u and $\tilde{u}$ are related by

$$\widetilde{\mu}_n = \mu_n\varphi_0 + \mu_{n+1}\varphi_1 + \mu_{n+2}\varphi_2, \quad n \geq 0. \tag{5}$$

One of the characterizations of the classical scalar orthogonal polynomials is that they are the only sequences of orthogonal polynomials whose derivatives are also sequences of orthogonal polynomials. The following proposition is the starting point to prove a similar result for the family $\mathcal{P}_{2,1}$.

Proposition 3.1. *Let u be a $\mathcal{P}_{2,1}$-functional such that $\Delta_0, \Delta_1, \dots, \Delta_n$ are non singular. Then, the corresponding finite segment $(P_k)_{k=0}^n$ of monic MOP satisfies*

$$\langle x^j P_k', \widetilde{u}\rangle = 0, \qquad j = 0, \dots, k-2, \quad k = 2, \dots, n,$$
$$\langle x^{k-1} P_k', \widetilde{u}\rangle = -E_k(\psi_1 + (k-1)\varphi_2), \qquad k = 1, \dots, n.$$

Proof. From the distributional equation $D(u\Phi) = u\Psi$ we have

$$\langle x^j P_k, D(u\Phi)\rangle = \langle x^j P_k, u\Psi\rangle,$$

or, equivalently,

$$-j\langle x^{j-1} P_k, u\Phi\rangle - \langle x^j P_k', u\Phi\rangle = \langle x^j P_k, u\Psi\rangle,$$

which, for $j = 0, \dots, k-1$, gives the result. $\square$

Corollary 3.2. *Under the conditions of Proposition 3.1, $(P_k')_{k=1}^n$ is a finite segment of MOP with respect to $\widetilde{u}$ if and only if the matrix $\psi_1 + (k-1)\varphi_2$ is non-singular for $k = 1, \dots, n$.*

The above corollary shows the interest in finding conditions that ensure the non-singularity of the matrices $\psi_1 + k\varphi_2$, $k = 0, 1, 2 \dots$. The next lemmas study the relation between the non-singularity of Δ_j, $j = 0, 1, \dots, p$, and $\psi_1 + k\varphi_2$, $k = 0, 1, \dots, q$, for small values of p and q. They also inform about the non-singularity of $\widetilde{\Delta}_k$, $k = 0, 1, \dots, q$, a result of interest since, in the scalar case, $\tilde{u}$ is quasi-definite for any classical functional u.

Lemma 3.3. *Let u be a $\mathcal{P}_{2,1}$-functional with $\Delta_0, \Delta_1, \Delta_2$ non-singular. Then, ψ_1 and $\widetilde{\Delta}_0$ are non-singular.*

Proof. If ψ_1 is singular, there exists $v \in \mathbb{C}^m \setminus \{0\}$ such that $\psi_1 v = 0$. Relation (4) for $n = 0$ gives $\mu_0\psi_0 + \mu_1\psi_1 = 0$. The non-singularity of $\mu_0 = \Delta_0$ implies $\psi_0 v = 0$. So, from (4) we have

$$\mu_{n-1}\varphi_0 v + \mu_n\varphi_1 v + \mu_{n+1}\varphi_2 v = 0, \qquad n \geq 1,$$

and, hence,

$$\Delta_2 \begin{pmatrix} \varphi_0 v \\ \varphi_1 v \\ \varphi_2 v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Also, $(\varphi_0 v, \varphi_1 v, \varphi_2 v) \neq (0, 0, 0)$ because $\det \Phi \neq 0$. Now, we can conclude the singularity of Δ_2, with contradicts the hypothesis. So, ψ_1 is non-singular.

On the other hand, the calculation of E_1 gives $E_1 = \mu_2 - \mu_1\mu_0^{-1}\mu_1$, which, according to Proposition 2.3, is non-singular because Δ_1 is non singular too. From (5) for $n = 0$ we get

$$\widetilde{\mu}_0 = \mu_0\varphi_0 + \mu_1\varphi_1 + \mu_2\varphi_2,$$

and (4) for $n = 0, 1$ gives

$$\mu_0\psi_0 + \mu_1\psi_1 = 0, \qquad \mu_0\varphi_0 + \mu_1\varphi_1 + \mu_2\varphi_2 = -(\mu_1\psi_0 + \mu_2\psi_1).$$

Therefore,

$$\widetilde{\Delta}_0 = \widetilde{\mu}_0 = -\mu_1\psi_0 - \mu_2\psi_1 = -(\mu_2 - \mu_1\mu_0^{-1}\mu_1)\psi_1 = -E_1\psi_1$$

is non-singular. □

As a first consequence, we obtain the following announced result.

Theorem 3.4. *If $u \in \mathcal{P}_{2,1}$ and $\Delta_0, \Delta_1, \Delta_2$ are non-singular, the $\mathbb{C}^{(m,m)}$-right-module $\mathcal{M}_{2,1}(u)$ is cyclic.*

Proof. Let us suppose that $D(u\Phi^{(i)}) = u\Psi^{(i)}$ with $\Phi^{(i)} \in \mathbb{P}_2^{(m)}$, $\Psi^{(i)} \in \mathbb{P}_1^{(m)}$ for $i = 1, 2$, and assume that $\det \Phi^{(1)} \neq 0$. We are going to prove that $\Phi^{(2)} = \Phi^{(1)}\Lambda$, $\Lambda \in \mathbb{C}^{(m,m)}$. Let $\Psi^{(i)}(x) = \psi_0^{(i)} + \psi_1^{(i)}x$ with $\psi_0^{(i)}, \psi_1^{(i)} \in \mathbb{C}^{(m,m)}$. Since u satisfies the hypothesis of Lemma 3.3, $\psi_1^{(1)}$ is non-singular. Hence, $A = \Phi^{(1)}(\psi_1^{(1)})^{-1}\psi_1^{(2)} - \Phi^{(2)}$ satisfies

$$D(uA) = u\left(\psi_0^{(1)}(\psi_1^{(1)})^{-1}\psi_1^{(2)} - \psi_0^{(2)}\right).$$

From (4) for $n = 0$, $\psi_0^{(i)} = -\mu_0^{-1}\mu_1\psi_1^{(i)}$, therefore, $D(uA) = 0$. If $A(x) = A_0 + A_1x + A_2x^2$, $A_i \in \mathbb{C}^{(m,m)}$, we get $\mu_nA_0 + \mu_{n+1}A_1 + \mu_{n+2}A_2 = 0$ for $n \geq 0$, which implies

$$\Delta_2 \begin{pmatrix} A_0 \\ A_1 \\ A_2 \end{pmatrix} = 0.$$

Since Δ_2 is non-singular, $A = 0$ and, thus, $\Phi^{(2)} = \Phi^{(1)}(\psi_1^{(1)})^{-1}\psi_1^{(2)}$. □

Now, we are going to consider $\mathcal{P}_{2,1}$-functionals satisfying the hypothesis of Lemma 3.3. In such a case we can write $\psi_1 = I$ without loss of generality because the Pearson-type equation can be written as $D(u\Phi\psi_1^{-1}) = u\Psi\psi_1^{-1}$.

Lemma 3.5. *Let u be a $\mathcal{P}_{2,1}$-functional with Δ_k non-singular for $k = 0, 1, 2, 3$. Then,*

(i) ψ_1 and $\psi_1 + \varphi_2$ are non-singular.

(ii) $\widetilde{\Delta}_0$ and $\widetilde{\Delta}_1$ are non-singular.

(iii) $\tilde{u}$ *is a* $\mathcal{P}_{2,1}$*-functional, that is,* $D(\widetilde{u}\widetilde{\Phi}) = \widetilde{u}\widetilde{\Psi}$*, with* $\widetilde{\Phi}(x) = \sum_{i=0}^{2} \tilde{\varphi}_i x^i$*,* $\widetilde{\Psi}(x) = \sum_{j=0}^{1} \tilde{\psi}_j x^j$*, where* $\tilde{\varphi}_i, \tilde{\psi}_j \in \mathbb{C}^{(m,m)}$ *and* $\det \tilde{\Phi} \neq 0$*. Moreover,* $\tilde{\Phi}$*,* $\tilde{\Psi}$ *can be chosen such that* $\tilde{\varphi}_2 = \psi_1^{-1}\varphi_2$ *and* $\tilde{\psi}_1 = \psi_1^{-1}(\psi_1 + 2\varphi_2)$*.*

Proof. We will assume without of loss of generality that $\psi_1 = I$.

(i) Let us suppose that $I + \varphi_2$ is singular. There exists $v \in \mathbb{C}^m \setminus \{0\}$ such that $\varphi_2 v = -v$. Writing (4) for $n = 0, 1$,

$$\mu_1 + \mu_0\psi_0 = 0, \qquad \mu_1(\psi_0 + \varphi_1)v + \mu_0\varphi_0 v = 0.$$

Then,

$$-\psi_0(\psi_0 + \varphi_1)v + \varphi_0 v = 0. \tag{6}$$

Consider (4) again, but for n and $n+1$:

$$\begin{cases} n\mu_{n-1}\varphi_0 + \mu_n(\psi_0 + n\varphi_1) + \mu_{n+1}(I + n\varphi_2) = 0, \\ (n+1)\mu_n\varphi_0 + \mu_{n+1}[\psi_0 + (n+1)\varphi_1] + \mu_{n+2}[I + (n+1)\varphi_2] = 0. \end{cases}$$

Multiplying the first equation on the right by $\psi_0 + \varphi_1$ and subtracting the second one, gives

$$\begin{aligned} n\mu_{n-1}\varphi_0(\psi_0 + \varphi_1) + \mu_n\left[\psi_0\left(\psi_0 + \varphi_1\right) - \varphi_0 + n\left(\varphi_1\left(\psi_0 + \varphi_1\right) - \varphi_0\right)\right] + \\ + n\mu_{n+1}\left[\varphi_2(\psi_0 + \varphi_1) - \varphi_1\right] - \mu_{n+2}\left[I + (n+1)\varphi_2\right] = 0. \end{aligned}$$

Then, taking into account (6), we get

$$\begin{aligned} \mu_{n-1}\varphi_0\left(\psi_0 + \varphi_1\right)v + \mu_n\left[\varphi_1\left(\psi_0 + \varphi_1\right) - \varphi_0\right]v + \\ + \ \mu_{n+1}\left[\varphi_2\left(\psi_0 + \varphi_1\right) - \varphi_1\right]v - \mu_{n+2}v = 0, \quad n \geq 1, \end{aligned} \tag{7}$$

which implies

$$\Delta_3 \begin{pmatrix} \varphi_0(\psi_0 + \varphi_1)v \\ [\varphi_1(\psi_0 + \varphi_1) - \varphi_0]v \\ [\varphi_2(\psi_0 + \varphi_1) - \varphi_1]v \\ -v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

This contradicts the non-singularity of Δ_3.

(ii) By Proposition 3.1 and Corollary 3.2, $\{P_1', P_2'\}$ is a finite segment of MOP with respect to $\widetilde{u}$. The result follows from Proposition 2.3.

(iii) The existence of matrix polynomials $\widetilde{\Phi}, \widetilde{\Psi}$ satisfying $D(\tilde{u}\tilde{\Phi}) = \tilde{u}\tilde{\Psi}$ is ensured if

$$\Psi\widetilde{\Phi} + \Phi\widetilde{\Phi}' = \Phi\widetilde{\Psi}. \tag{8}$$

Writing $\widetilde{\Phi}(x) = \widetilde{\varphi}_0 + \widetilde{\varphi}_1 x + \widetilde{\varphi}_2 x^2$, $\widetilde{\Psi}(x) = \widetilde{\psi}_0 + \widetilde{\psi}_1 x$, (8) is equivalent to the system

$$\begin{pmatrix} \psi_0 & 0 & \varphi_0 & 0 \\ I & \psi_0 & \varphi_1 & 0 \\ 0 & I & \varphi_2 & 0 \\ 0 & 0 & 0 & I+2\varphi_2 \end{pmatrix} \begin{pmatrix} \widetilde{\varphi}_0 \\ \widetilde{\varphi}_1 \\ \widetilde{\varphi}_1 - \widetilde{\psi}_0 \\ \widetilde{\varphi}_2 \end{pmatrix} = \begin{pmatrix} 0 \\ \varphi_0(\widetilde{\psi}_1 - 2\widetilde{\varphi}_2) \\ \varphi_1(\widetilde{\psi}_1 - 2\widetilde{\varphi}_2) - \psi_0\widetilde{\varphi}_2 \\ \varphi_2\widetilde{\psi}_1 \end{pmatrix}. \tag{9}$$

A solution of the last equation is $\widetilde{\psi}_1 = I + 2\varphi_2$, $\widetilde{\varphi}_2 = \varphi_2$. With this choice, converting the system into triangular form gives

$$\begin{pmatrix} I & \psi_0 & \varphi_1 \\ 0 & I & \varphi_2 \\ 0 & 0 & \varphi_0 - \psi_0\varphi_1 + \psi_0^2\varphi_2 \end{pmatrix} \begin{pmatrix} \widetilde{\varphi}_0 \\ \widetilde{\varphi}_1 \\ \widetilde{\varphi}_1 - \widetilde{\psi}_0 \end{pmatrix} = \begin{pmatrix} \varphi_0 \\ \varphi_1 - \psi_0\varphi_2 \\ -\psi_0(\varphi_0 - \psi_0\varphi_1 + \psi_0^2\varphi_2) \end{pmatrix}.$$

From (4) for $n = 0$, $\mu_0\psi_0 + \mu_1 = 0$, so,

$$\Upsilon := \varphi_0 - \psi_0\varphi_1 + \psi_0^2\varphi_2 = \varphi_0 + \mu_0^{-1}\mu_1\varphi_1 + (\mu_0^{-1}\mu_1)^2\varphi_2 =$$

$$= \mu_0^{-1}(\mu_0\varphi_0 + \mu_1\varphi_1 + \mu_1\mu_0^{-1}\mu_1\varphi_2).$$

Since $E_1 = \mu_2 - \mu_1\mu_0^{-1}\mu_1$,

$$\Upsilon = \mu_0^{-1}(\mu_0\varphi_0 + \mu_1\varphi_1 + \mu_2\varphi_2 - E_1\varphi_2)$$

that, keeping in mind (4) for $n = 1$, can be expressed as

$$\Upsilon = -\mu_0^{-1}(\mu_1\psi_0 + \mu_2 + E_1\varphi_2) =$$

$$= -\mu_0^{-1}(-\mu_1\mu_0^{-1}\mu_1 + \mu_2 + E_1\varphi_2) = -\mu_0^{-1}E_1(I + \varphi_2).$$

That is, Υ is non-singular, what ensures that (9) has a solution.

Finally, we are going to prove that $\det\widetilde{\Phi} \neq 0$. From (8) we can deduce

$$\Phi(\tilde{\Psi} - \tilde{\Phi}') = \Psi\tilde{\Phi}.$$

Since $\det\Phi \neq 0$, $\det\tilde{\Phi} = 0$ implies $\det(\tilde{\Psi} - \tilde{\Phi}') = 0$. However, taking into account that $\tilde{\psi}_1 = I + 2\varphi_2$ and $\tilde{\varphi}_2 = \varphi_2$ we get $\tilde{\Psi}(x) - \tilde{\Phi}'(x) = \tilde{\psi}_0 - \tilde{\varphi}_1 + Ix$, which has non-null determinant. $\square$

Lemma 3.6. *Let u be a $\mathcal{P}_{2,1}$-functional with Δ_k non-singular for $k = 0, 1, 2, 3, 4$. Then,*

(i) $\psi_1 + j\varphi_2$ *is non-singular for* $j = 0, 1, 2$.

(ii) $\widetilde{\Delta}_j$ *is non-singular for* $j = 0, 1, 2$.

Proof. We will assume without of loss of generality that $\psi_1 = I$.

(i) Taking into account Lemma 3.5 *(iii)*, the functional $\widetilde{u}$ satisfies $D(\widetilde{u}\widetilde{\Phi}) = \widetilde{u}\widetilde{\Psi}$, with $\widetilde{\varphi}_2 = \varphi_2$, $\widetilde{\psi}_1 = I + 2\varphi_2$, where $\tilde{\varphi}_i$, $\tilde{\psi}_j$ have the same meaning as in the proof of the previous lemma.

Let us suppose that $I + 2\varphi_2$ is singular. Then, there exists $v \in \mathbb{C}^m \setminus \{0\}$ such that $\varphi_2 v = -\frac{1}{2}v$, that is, $\widetilde{\psi}_1 v = 0$. Since $D(\widetilde{u}\widetilde{\Phi}) = \widetilde{u}\widetilde{\Psi}$, we have

$$n(\widetilde{\mu}_{n-1}\widetilde{\varphi}_0 + \widetilde{\mu}_n\widetilde{\varphi}_1 + \widetilde{\mu}_{n+1}\widetilde{\varphi}_2) = -(\widetilde{\mu}_n\widetilde{\psi}_0 + \widetilde{\mu}_{n+1}\widetilde{\psi}_1), \qquad n \geq 0,$$

which, for $n = 0$, gives $\widetilde{\mu}_0\widetilde{\psi}_0 + \widetilde{\mu}_1\widetilde{\psi}_1 = 0$. Hence, $\widetilde{\psi}_0 v = 0$ because, from Lemma 3.3, $\widetilde{\mu}_0 = \tilde{\Delta}_0$ is non-singular. So,

$$(\widetilde{\mu}_{n-1}\widetilde{\varphi}_0 + \widetilde{\mu}_n\widetilde{\varphi}_1 + \widetilde{\mu}_{n+1}\widetilde{\varphi}_2)v = 0, \quad n \geq 1. \tag{10}$$

According to (5),

$$\begin{aligned}\mu_{n-1}\varphi_0\widetilde{\varphi}_0 v + \mu_n(\varphi_1\widetilde{\varphi}_0 + \varphi_0\widetilde{\varphi}_1)v + \mu_{n+1}(\varphi_2\widetilde{\varphi}_0 + \varphi_1\widetilde{\varphi}_1 + \varphi_0\widetilde{\varphi}_2)v + \\ + \mu_{n+2}(\varphi_2\widetilde{\varphi}_1 + \varphi_1\widetilde{\varphi}_2)v + \mu_{n+3}\varphi_2\widetilde{\varphi}_2 v = 0, \quad n \geq 1,\end{aligned}$$

and from here we can deduce the singularity of Δ_4, because $\varphi_2\widetilde{\varphi}_2 v = \varphi_2^2 v = \frac{1}{4}v \neq 0$. This contradicts the hypothesis. So, $\widetilde{\psi}_1$ is non-singular.

(ii) From Corollary 3.2, $\{P_1', P_2', P_3'\}$ is a finite segment of MOP with respect to $\widetilde{u}$ and, so, Proposition 2.3 ensures that $\widetilde{\Delta}_2$ is non-singular. □

The previous lemmas can be generalized through an inductive process. This process will need the following result too.

Lemma 3.7. *Let* $u \in \mathbb{P}^{(m)\prime}$ *and* $F \in \mathbb{P}_p^{(m)}$, *with* $\det F \neq 0$. *We denote* $\widetilde{u} = uF$ *and we suppose that there exist* $v_0, v_1, \ldots, v_q \in \mathbb{C}^m$, *with* $v_k \neq 0$ *for some* $k \in \{0, 1, \ldots, q\}$, *such that the moments* $(\widetilde{\mu}_n)_{n\geq 0}$ *of the functional* $\widetilde{u}$ *satisfy*

$$\sum_{j=0}^{q} \widetilde{\mu}_{n+j} v_j = 0, \qquad \forall n \geq 0.$$

Then, there exist $w_1, w_2, \ldots, w_{p+q} \in \mathbb{C}^m$, *with* $w_k \neq 0$ *for some* $k \in \{0, \ldots, p+q\}$, *such that the moments* $(\mu_n)_{n\geq 0}$ *of the functional* u *satisfy*

$$\sum_{k=0}^{p+q} \mu_{n+k} w_k = 0, \qquad \forall n \geq 0.$$

Proof. We will write $F(x) = f_0 + f_1 x + \cdots + f_p x^p$ with $f_i \in \mathbb{C}^{(m,m)}$. Then, $\widetilde{\mu}_n = \sum_{i=0}^{p} \mu_{n+i} f_i$ and the hypothesis of the lemma gives

$$0 = \sum_{j=0}^{q} \widetilde{\mu}_{n+j} v_j = \sum_{j=0}^{q} \left(\sum_{i=0}^{p} \mu_{n+j+i} f_i \right) v_j = \sum_{k=0}^{p+q} \mu_{n+k} \sum_{i=0}^{p} f_i v_{k-i},$$

with the convention $v_{-1} = \cdots = v_{-p} = 0$. So, the vectors $w_k = \sum_{i=0}^{p} f_i v_{k-i}$, $k = 0, \ldots, p+q$, satisfy the equality of the statement. It will be enough to prove that not all the vectors w_k are null. If all of them are zero, $\sum_{i=0}^{p} f_i v_{k-i} = 0$ for $k = 0, \ldots, p+q$, and this implies

$$0 = \sum_{k=0}^{p+q} x^k \sum_{i=0}^{p} f_i v_{k-i}, \qquad \forall x \in \mathbb{C},$$

or, equivalently,

$$0 = \sum_{j=0}^{q} x^j \left(\sum_{i=0}^{p} f_i x^i \right) v_j = F(x) \sum_{j=0}^{q} v_j x^j, \qquad \forall x \in \mathbb{C}.$$

Since $\det F \neq 0$, we obtain from Remark 2.8 that $\sum_{j=0}^{q} v_j x^j = 0$ for all $x \in \mathbb{C}$, which means that $v_j = 0$ for $j = 0, \ldots, q$, in contradiction with the hypothesis. □

Now we can reach the generalization of Lemmas 3.3, 3.5 and 3.6.

Theorem 3.8. *Let u be a $\mathcal{P}_{2,1}$-functional with Δ_k non-singular for $k = 0, 1, \ldots, n$, where $n \geq 2$. Then, $\psi_1 + j\varphi_2$ and $\tilde{\Delta}_j$ are non-singular for $j = 0, 1, \ldots, n-2$.*

Proof. Due to Lemmas 3.3, 3.5 and 3.6 the result is true for $n = 2, 3, 4$. We will assume the statement for an index $n \geq 2$, and we will prove that it is also true for $n + 1$.

Assume that $\Delta_0, \Delta_1, \ldots, \Delta_n, \Delta_{n+1}$ are non-singular. Then, the hypothesis of induction implies that $\psi_1 + j\varphi_2$ and $\tilde{\Delta}_j$ are non-singular for $j = 0, 1, \ldots, n-2$. We only must prove that $\psi_1 + (n-1)\varphi_2$ and $\tilde{\Delta}_{n-1}$ are non-singular too. For this purpose we will introduce a set of $\mathcal{P}_{2,1}$-functionals $u^{(j)}$, $j = 0, 1, \ldots$, using the superscript (j) for the associated elements.

Let us define $u^{(0)} = u$, $\Phi^{(0)} = \Phi$, $\Psi^{(0)} = \Psi$. Taking into account Lemmas 3.5 and 3.6, given $u^{(1)} = u^{(0)} \Phi^{(0)} (\psi_1^{(0)})^{-1}$ there exist $\Phi^{(1)} \in \mathbb{P}_2^{(m)}$, $\Psi^{(1)} \in \mathbb{P}_1^{(m)}$, satisfying $D\left(u^{(1)} \Phi^{(1)}\right) = u^{(1)} \Psi^{(1)}$, with $\det \Phi^{(1)} \neq 0$, $\varphi_2^{(1)} =$

$\varphi_2^{(0)}$ and $\psi_1^{(1)} = \psi_1^{(0)} + 2\varphi_2^{(0)}$ non-singular. Moreover, from Proposition 3.1, $E_k^{(1)} = -\frac{1}{k+1} E_{k+1}^{(0)}(\psi_1^{(0)} + k\varphi_2^{(0)})$. This implies that $E_0^{(1)}, \ldots, E_{n-2}^{(1)}$ and, thus, $\Delta_0^{(1)}, \ldots, \Delta_{n-2}^{(1)}$ are non-singular.

Following this procedure, we can construct inductively a set of $\mathcal{P}_{2,1}$-functionals $u^{(j)}$, $j = 0, 1, \ldots, l-1$ $\left(l = \left[\frac{n}{2}\right]\right)$, satisfying

$$u^{(j+1)} = u^{(j)} \Phi^{(j)} (\psi_1^{(j)})^{-1},$$

$$D(u^{(j)}\Phi^{(j)}) = u^{(j)}\Psi^{(j)}, \quad \varphi_2^{(j)} = \varphi_2, \quad \psi_1^{(j)} = \psi_1 + 2j\varphi_2,$$

$$E_k^{(j+1)} = -\frac{1}{k+1} E_{k+1}^{(j)} \left[\psi_1 + (2j+k)\varphi_2\right],$$

$$\Delta_0^{(j)}, \ldots, \Delta_{n-2j}^{(j)} \quad \text{non-singular.}$$

Let us suppose that n is even ($n = 2l$). Then, $\Delta_0^{(l-1)}, \Delta_1^{(l-1)}, \Delta_2^{(l-1)}$ are non-singular. If $\psi_1 + (n-1)\varphi_2 = \psi_1^{(l-1)} + \varphi_2^{(l-1)}$ is singular, the same arguments that lead to (4) in the proof of Lemma 3.5 give now

$$\sum_{j=0}^{3} \mu_{k+j}^{(l-1)} v_j = 0, \qquad v_3 \neq 0, \qquad k \geq 0.$$

Since $u^{(l-1)} = uF$, $\deg F \leq 2l - 2 = n - 2$, we get from Lemma 3.7

$$\sum_{j=0}^{n+1} \mu_{k+j} w_j = 0, \qquad \text{some } w_j \neq 0, \qquad k \geq 0.$$

This contradicts the non-singularity of Δ_{n+1}, so, $\psi_1 + (n-1)\varphi_2$ must be non-singular.

If, on the contrary, n is odd ($n = 2l+1$), $\Delta_0^{(l-1)}, \Delta_1^{(l-1)}, \Delta_2^{(l-1)}, \Delta_3^{(l-1)}$ are non-singular. Thus, analogously to (8) in the proof of Lemma 3.6, we find that, if $\psi_1 + (n-1)\varphi_2 = \psi_1^{(l-1)} + 2\varphi_2^{(l-1)}$ is singular,

$$\sum_{j=0}^{4} \mu_{k+j}^{(l-1)} v_j = 0, \qquad v_4 \neq 0, \qquad k \geq 0.$$

Now, $u^{(l-1)} = uF$, $\deg F \leq 2l - 2 = n - 3$, so, Lemma 3.7 gives again the same condition

$$\sum_{j=0}^{n+1} \mu_{k+j} w_j = 0, \qquad \text{some } w_j \neq 0, \qquad k \geq 0,$$

so, $\psi_1 + (n-1)\varphi_2$ is also non-singular in this case.

Finally, the non-singularity of $\tilde{\Delta}_{n-1}$ follows from Proposition 2.3 and the relation $\tilde{E}_{n-1} = -\frac{1}{n} E_n(\psi_1 + (n-1)\varphi_2)$ given in Proposition 3.1. □

The previous theorem and Corollary 3.2 have the following immediate consequences.

Corollary 3.9. *If u is a quasi-definite $\mathcal{P}_{2,1}$-functional, then $\psi_1 + n\varphi_2$ is non-singular for $n = 0, 1, 2, \ldots$.*

Corollary 3.10. *If u is a quasi-definite $\mathcal{P}_{2,1}$-functional, then $\widetilde{u} = u\Phi$ is a quasi-definite $\mathcal{P}_{2,1}$-functional too. Moreover, if $(P_n)_{n\geq 0}$ is the sequence of monic MOP with respect to u, then $\left(\frac{1}{n} P'_n\right)_{n\geq 1}$ is the sequence of monic MOP with respect to $\tilde{u}$.*

Remark 3.11. The Pearson-type equation $D(u\Phi) = u\Psi$, $\Phi \in \mathbb{P}_2^{(m)}$, $\Psi \in \mathbb{P}_1^{(m)}$, is equivalent to the recurrence $n\mu_{n-1}\varphi_0 + \mu_n(\psi_0 + n\varphi_1) + \mu_{n+1}(\psi_1 + n\varphi_2) = 0$, $n \geq 0$. Therefore, the non-singularity of the matrices $\psi_1 + n\varphi_2$ for $n \geq 0$ is a sufficient condition for the existence of a solution u of the Pearson-type equation. Indeed, this condition ensures that the solutions are determined by $\mu_0 = \langle I, u \rangle$ or, in other words, the solution is unique up to left matrix factors. Then, according to Corollary 3.9, if the Pearson-type equation has a quasi-definite solution, the quasi-definite solutions are exactly those solutions determined by a non-singular matrix μ_0.

3.1 Characterization of the family $\mathcal{P}_{2,1}$

In the scalar case, the classical orthogonal polynomials can be characterized alternatively by a Pearson-type equation (see [8, 19, 20, 22]), the orthogonality of the derivatives (see [3, 8, 17, 19, 20]) or a linear relation between the polynomials P_n and P'_{n+1}, P'_n, P'_{n-1} (see [18]). The consequences of the previous analysis provide an analogue of these equivalences for the matrix case, which constitute a characterization of the quasi-definite $\mathcal{P}_{2,1}$-functionals. In the proof of this characterization we will need the following results too.

Lemma 3.12. *Let $u \in \mathbb{P}^{(m)'}$ such that Δ_n is non-singular. Then,*

$$uP = 0, \;\; P \in \mathbb{P}_n^{(m)} \quad \Rightarrow \quad P = 0.$$

Proof. Let $P(x) = \sum_{i=0}^{n} A_i x^i$, $A_i \in \mathbb{C}^{(m,m)}$. Then, $uP = 0$ is equivalent to $\mu_k A_0 + \cdots + \mu_{k+n} A_n = 0$ for $k \geq 0$, which implies

$$\Delta_n \begin{pmatrix} A_0 \\ \vdots \\ A_n \end{pmatrix} = 0,$$

and, thus, $P = 0$ if Δ_n is non-singular. $\square$

Proposition 3.13. *Let $u, v \in \mathbb{P}^{(m)'}$ with u quasi-definite and (P_n) its corresponding sequence of monic MOP. Then, the following statements are equivalent:*

(i) $v = uA, \quad A \in \mathbb{P}_p^{(m)}$.

(ii) (P_n) *is quasi-orthogonal of order not greater than p with respect to v:*

$\langle x^k P_n, v\rangle = 0, \quad k = 0, \ldots, n-p-1.$

Proof. See [5]. $\square$

Here is the referred characterization of the quasi-definite $\mathcal{P}_{2,1}$-functionals.

Theorem 3.14. *Let $u \in \mathbb{P}^{(m)'}$ be quasi-definite and (P_n) its sequence of monic MOP. Then, the following assertions are equivalent:*

(i) u is a $\mathcal{P}_{2,1}$-functional.

(ii) (P'_n) is a sequence of MOP with respect to a quasi-definite functional $\widetilde{u}$.

(iii) There exist matrices $a_n, b_n \in \mathbb{C}^{(m,m)}$ such that

$$P_n = \frac{1}{n+1}P'_{n+1} + a_n P'_n + b_n P'_{n-1}, \qquad n \geq 0,$$

with $\gamma_n - b_n$ non-singular for $n \geq 1$.

Moreover, $\widetilde{u} = u\Phi$, $\Phi \in \mathbb{P}_2^{(m)}$, $\det\Phi \neq 0$ and $D(u\Phi) = u\Psi$, $\Psi \in \mathbb{P}_1^{(m)}$. Besides, $\widetilde{u}$ is a quasi-definite $\mathcal{P}_{2,1}$-functional too.

Proof.

(ii) $\Leftrightarrow$ *(iii)* The sequence of matrix polynomials (P_n) satisfies the recurrence relation,

$$xP_n = P_{n+1} + \beta_n P_n + \gamma_n P_{n-1},$$

so,

$$P_n = -xP'_n + P'_{n+1} + \beta_n P'_n + \gamma_n P'_{n-1}. \tag{11}$$

If we assume *(ii)*, (P'_n) also satisfies a recurrence relation

$$\frac{1}{n}xP'_n = \frac{1}{n+1}P'_{n+1} + \frac{1}{n}\tilde{\beta}_{n-1}P'_n + \frac{1}{n-1}\tilde{\gamma}_{n-1}P'_{n-1} \tag{12}$$

and, then, (11) and (12) imply

$$P_n = \frac{1}{n+1}P'_{n+1} + a_n P'_n + b_n P'_{n-1}, \tag{13}$$

where $a_n = \beta_n - \tilde{\beta}_{n-1}$ and $b_n = \gamma_n - \frac{n}{n-1}\tilde{\gamma}_{n-1}$. Notice that $\gamma_n - b_n = \frac{n}{n-1}\tilde{\gamma}_{n-1}$ is non-singular.

For the converse, from (11) and (13),

$$\frac{1}{n} x P_n' = \frac{1}{n+1} P_{n+1}' + \frac{1}{n}\left(\beta_n - a_n\right) P_n' + \frac{1}{n}\left(\gamma_n - b_n\right) P_{n-1}'.$$

Now, we have a recurrence relation for (P_n') with $\tilde{\beta}_{n-1} = \beta_n - a_n$ and $\tilde{\gamma}_{n-1} = \frac{n-1}{n}(\gamma_n - b_n)$. Since $\gamma_n - b_n$ is non-singular, the Favard theorem assures the existence of a functional $\tilde{u} \in \mathbb{P}^{(m)'}$ such that (P_n') is a sequence of MOP with respect to $\tilde{u}$.

(ii), (iii) $\Rightarrow$ *(i)* Assume the relation $P_n = \frac{1}{n+1} P_{n+1}' + a_n P_n' + b_n P_{n-1}'$ and the fact that (P_n') is a sequence of MOP with respect to a certain functional $\tilde{u}$. Notice that this last hypothesis implies the non-singularity of $\tilde{E}_{n-1} = \frac{1}{n}\langle x^{n-1} P_n', \tilde{u}\rangle$ for $n \geq 1$. Under the assumptions,

$$\langle x^k P_n, \tilde{u}\rangle = 0, \qquad k = 0, \ldots, n-3.$$

So, (P_n) is a quasi-orthogonal sequence with respect to $\tilde{u}$ of order not greater than 2. Proposition 3.13 says that there exists $\Phi \in \mathbb{P}_2^{(m)}$ such that $\tilde{u} = u\Phi$. Setting $w = D(u\Phi)$,

$$\langle x^k P_n, w\rangle = -k\langle x^{k-1} P_n, u\Phi\rangle - \langle x^k P_n', u\Phi\rangle = 0, \quad k = 0, \ldots, n-2.$$

Hence, (P_n) is quasi-orthogonal with respect to w of order not greater than 1 and, thus, there exists $\Psi \in \mathbb{P}_1^{(m)}$ such that $w = u\Psi$.

It only remains to prove that $\det \Phi \neq 0$. For this purpose, notice that the equality

$$\langle x^{n-1} P_n, D(u\Phi)\rangle = \langle x^{n-1} P_n, u\Psi\rangle$$

gives

$$-(n-1) E_n \varphi_2 - n\tilde{E}_{n-1} = E_n \psi_1.$$

Hence, $\psi_1 + (n-1)\varphi_2$ is non-singular for $n \geq 1$. Suppose $\det \Phi = 0$. Then, according to Remark 2.8, there exists $v \in \mathbb{C}^m[x] \setminus \{0\}$ such that $\Phi v = 0$. Consider the matrix polynomial $A \in \mathbb{P}^{(m)}$ whose columns are all equal to v. Taking into account Lemma 3.12, the equality

$$u(\Psi - \Phi')A = (Du)\Phi A = 0$$

proves that $(\Psi - \Phi')v = 0$. So, $\Psi v + \Phi v' = 0$ and, if $v(x) = v_0 + \cdots + v_n x^n$, $v_i \in \mathbb{C}^m$, with $v_n \neq 0$, we get $(\psi_1 + n\varphi_2)v_n = 0$, which is impossible.

(i) $\Rightarrow$ *(ii)* This implication is given by Corollary 3.10. $\square$

Remark 3.15. Theorem 3.14 ensures that any quasi-definite $\mathcal{P}_{2,1}$-functional u generates a sequence $(u^{(n)})_{n\geq 0}$ of quasi-definite $\mathcal{P}_{2,1}$-functionals, starting with $u^{(0)} = u$, and such that, for $n \geq 0$,

$$u^{(n+1)} = u^{(n)}\Phi^{(n)}, \quad \Phi^{(n)} \in \mathbb{P}_2^{(m)}, \quad \det \Phi^{(n)} \neq 0,$$

$$D(u^{(n)}\Phi^{(n)}) = u^{(n)}\Psi^{(n)}, \quad \Psi^{(n)} \in \mathbb{P}_1^{(m)}.$$

Moreover, the k-th derivatives $(P_n^{(k)})_{n\geq k}$ form a sequence of MOP with respect to $u^{(k)}$. That is, as in the scalar case, if the first derivatives of a sequence of MOP are orthogonal, the higher order derivatives are orthogonal too.

Remark 3.16. If u is not quasi-definite but $\Delta_0, \ldots, \Delta_n$ are non-singular, *(ii)* and *(iii)* remain equivalent, but for the finite segment $(P_k)_{k=0}^{n}$ of monic MOP with respect to u. Besides, in this case, *(i)* also implies *(ii)* and *(iii)*, but only for the finite segment $(P_k)_{k=0}^{n-1}$, according to Theorem 3.8.

The following consequence of Theorem 3.14 will be of interest when studying the differential equation associated with the zero class MOP.

Corollary 3.17. *If a sequence (P_n) of monic MOP belongs to the family $\mathcal{P}_{2,1}$, then $P'_{n\pm 1} \in \operatorname{span}_{\mathbb{C}^{(m,m)}}\{xP'_n, P'_n, P_n\}$. More precisely,*

$$P'_{n-1} = E_{n-1}M_{n-2}M_{2n-1}^{-1}E_n^{-1}\left\{\left(x + \frac{1}{n}\pi_n\right)P'_n - nP_n\right\},$$

$$P'_{n+1} = (n+1)E_n\left\{\left(\varphi_2 M_{2n-1}^{-1}E_n^{-1}x - \frac{1}{n}M_{2n-2}M_{2n-1}^{-1}E_n^{-1}\pi_n + \right.\right.$$
$$\left.\left. + \frac{1}{n+1}E_n^{-1}\pi_{n+1}\right)P'_n + M_{n-1}M_{2n-1}^{-1}E_n^{-1}P_n\right\},$$

where $E_n = \langle x^n P_n, u\rangle$, $P_n(x) = x^n + \pi_n x^{n-1} + \cdots$ and $M_n = \psi_1 + n\varphi_2$.

Proof. Using (11) and (13), we get by eliminating P'_{n+1} and P'_{n-1} respectively,

$$\begin{cases} nP_n = (x - \beta_n + (n+1)a_n)\,P'_n - (\gamma_n - (n+1)b_n)\,P'_{n-1}, \\ \left(1 - b_n\gamma_n^{-1}\right)P_n = \left(\frac{1}{n+1} - b_n\gamma_n^{-1}\right)P'_{n+1} + \left(b_n\gamma_n^{-1}(x-\beta_n) + a_n\right)P'_n. \end{cases}$$

The matrix coefficients $\beta_n, \gamma_n, \tilde{\beta}_n, \tilde{\gamma}_n, a_n, b_n$ can be expressed in terms of E_n and π_n since

$$\begin{aligned} \beta_n &= \pi_n - \pi_{n+1}, & \gamma_n &= E_n E_{n-1}^{-1}, \\ \tilde{\beta}_{n-1} &= \tfrac{n-1}{n}\pi_n - \tfrac{n}{n+1}\pi_{n+1}, & \tilde{\gamma}_{n-1} &= \tfrac{n-1}{n} E_n M_{n-1} M_{n-2}^{-1} E_{n-1}^{-1}, \\ a_n &= \beta_n - \tilde{\beta}_{n-1}, & b_n &= \gamma_n - \tfrac{n}{n-1}\tilde{\gamma}_{n-1}. \end{aligned}$$

From here, it is just a matter calculation to get the result, using the fact that $M_k M_j^{-1} = \hat{M}_k \hat{M}_j^{-1} = \hat{M}_j^{-1}\hat{M}_k$, where $\hat{M}_n = I + n\varphi_2\psi_1^{-1}$. □

3.2 Examples

The purpose of the following examples is to show that non-diagonalizable matrix $\mathcal{P}_{2,1}$-functionals do exist, even in the positive definite case, and that the family $\mathcal{P}_{2,1}$ is strictly bigger than the zero class (excepting the scalar case). Indeed, the presented examples are all positive definite and lie on the class $s = 1$. The matrix functionals of the examples have the structure $w(x)R(x)\,dx$, where w is a classical scalar weight and

$$R = \begin{pmatrix} p + qq^* & bq \\ \bar{b}q^* & |b|^2 \end{pmatrix}, \quad p, q \in \mathbb{P},$$

$$p \text{ with positive leading coefficient}, \quad \deg q = 1, \quad b \in \mathbb{C} \setminus \{0\}.$$

We will deal with a canonical form of these functionals, since any of them is congruent to one with the form

$$w(x) \begin{pmatrix} \hat{p}(x) + |a|^2 x^2 & ax \\ \bar{a}x & 1 \end{pmatrix} dx, \quad \hat{p} \in \mathbb{P} \text{ monic}, \quad a \in \mathbb{C} \setminus \{0\}.$$

This kind of functionals are never diagonalizable by congruence, neither by equivalence. This is a consequence of the fact that, as can be easily checked, any functional $W(x)\,dx$, with

$$W = \begin{pmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{pmatrix},$$

is non-diagonalizable by equivalence if $\{w_{11}, w_{12}, w_{22}\}$ is linearly independent and $\{w_{12}, w_{21}\}$ is linearly dependent.

Example 2. Let $u \in \mathbb{P}^{(2)'}$ given by

$$u = e^{-x^2} \begin{pmatrix} 1 + |a|^2 x^2 & ax \\ \bar{a}x & 1 \end{pmatrix} dx, \quad x \in \mathbb{R}, \quad a \in \mathbb{C} \setminus \{0\}.$$

It is not a zero class functional, but its class is $s=1$ due to the equality

$$Du = u \begin{pmatrix} (|a|^2-2)x & a \\ \bar{a}(1-|a|^2x^2) & -(|a|^2+2)x \end{pmatrix}.$$

Besides, it is a $\mathcal{P}_{2,1}$-functional with $\mathcal{M}_{2,1}(u) = \operatorname{span}_{\mathbb{C}^{(2,2)}}\{\Phi\}$, where

$$\Phi(x) = \begin{pmatrix} |a|^2+2 & 0 \\ -\bar{a}|a|^2x & 1 \end{pmatrix}.$$

The corresponding Pearson-type equation is $D(u\Phi) = u\Psi$, with

$$\Psi(x) = \begin{pmatrix} -4x & a \\ 2\bar{a} & -(|a|^2+2)x \end{pmatrix}.$$

Any right multiple of Φ by a non-singular matrix factor can be chosen as a generator of $\mathcal{M}_{2,1}(u)$, therefore, it will play a similar role in the Pearson-type equation for u. However, if we choose

$$\Phi^{(0)} = \Phi \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix},$$

the new functional $u^{(1)} = u\Phi^{(0)}$ is again a positive definite $\mathcal{P}_{2,1}$-functional of similar type. Indeed,

$$u^{(1)} = e^{-x^2} \begin{pmatrix} |a|^2+2+2|a|^2x^2 & 2ax \\ 2\bar{a}x & 2 \end{pmatrix} dx, \quad x \in \mathbb{R}.$$

This shows explicitly in the present example the general fact that any quasi-definite $\mathcal{P}_{2,1}$-functional generates a sequence of $\mathcal{P}_{2,1}$-functionals, according to Theorem 3.14 and Remark 3.15.

Example 3. The functional $u \in \mathbb{P}^{(2)'}$ defined by

$$u = x^r e^{-x} \begin{pmatrix} x+|a|^2x^2 & ax \\ \bar{a}x & 1 \end{pmatrix} dx, \quad x \in (0,\infty), \quad a \in \mathbb{C}\setminus\{0\}, \quad r > -1,$$

lies again on the class $s=1$ since

$$D(uxI) = u \begin{pmatrix} r+2+(|a|^2-1)x & a \\ -\bar{a}|a|^2x^2 & r+1-(|a|^2+1)x \end{pmatrix}.$$

It is also a $\mathcal{P}_{2,1}$-functional, with $\mathcal{M}_{2,1}(u) = \operatorname{span}_{\mathbb{C}^{(2,2)}}\{\Phi\}$ generated by

$$\Phi(x) = \begin{pmatrix} (|a|^2+1)x & 0 \\ -\bar{a}|a|^2x^2 & x \end{pmatrix}.$$

The Pearson-type equation is $D(u\Phi) = u\Psi$, where

$$\Psi(x) = \begin{pmatrix} (r+2)(|a|^2+1) - x & a \\ -(r+2)\bar{a}|a|^2 x & r+1-(|a|^2+1)x \end{pmatrix}.$$

Notice that $u^{(1)} = u\Phi$ is given by

$$u^{(1)} = x^{r+1}e^{-x} \begin{pmatrix} (|a|^2+1)x + |a|^2x^2 & ax \\ \bar{a}x & 1 \end{pmatrix} dx, \quad x \in (0,\infty),$$

so, it is a positive definite $\mathcal{P}_{2,1}$-functional of similar type.

Example 4. The functional $u \in \mathbb{P}^{(2)'}$ given by

$$u = x^r e^{-x} \begin{pmatrix} x^2 + |a|^2x^2 & ax \\ \bar{a}x & 1 \end{pmatrix} dx, \quad x \in (0,\infty), \quad a \in \mathbb{C}\setminus\{0\}, \quad r > -1,$$

is also in the class $s = 1$ since

$$D(ux^2 I) = u \begin{pmatrix} (r+|a|^2+4)x - x^2 & a \\ -\bar{a}(|a|^2+1)x^2 & (r-|a|^2+2)x - x^2 \end{pmatrix},$$

and belongs to the family $\mathcal{P}_{2,1}$, with $\mathcal{M}_{2,1}(u) = \mathrm{span}_{\mathbb{C}^{(2,2)}}\{\Phi\}$ generated by

$$\Phi(x) = \begin{pmatrix} x & -a \\ 0 & (r+|a|^2+2)x \end{pmatrix}.$$

The Pearson-type equation is $D(u\Phi) = u\Psi$, with

$$\Psi(x) = \begin{pmatrix} (r+|a|^2+3) - x & a \\ -\bar{a}(|a|^2+1)x & (r+1)(r+2) - (r+|a|^2+2)x \end{pmatrix}.$$

As in the previous cases, there is a choice of $\Phi^{(0)} \in \mathcal{M}_{2,1}(u)$ that makes $u^{(1)} = u\Phi^{(0)}$ a positive definite $\mathcal{P}_{2,1}$-functional of similar type. The choice is

$$\Phi^{(0)} = \Phi \begin{pmatrix} r+1 & 0 \\ 0 & 1 \end{pmatrix},$$

and the new functional is then

$$u^{(1)} = x^{r+1}e^{-x} \begin{pmatrix} (r+1)(|a|^2+1)x^2 & (r+1)ax \\ (r+1)\bar{a}x & r+2 \end{pmatrix} dx, \quad x \in (0,\infty).$$

4 The zero class

The zero class is a specially simple subset of the family $\mathcal{P}_{2,1}$. This simplicity allows for zero class functionals a deeper analysis than for general $\mathcal{P}_{2,1}$-functionals. According to the definition of the zero class we suppose in this section that $u \in \mathbb{P}^{(m)\prime}$ is a quasi-definite functional that satisfies a Pearson-type equation

$$D\,(u\alpha I) = u\Psi, \quad \alpha \in \mathbb{P}_2 \setminus \{0\}, \quad \Psi \in \mathbb{P}_1^{(m)}. \tag{14}$$

We will use the notation $\alpha(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2$, $\alpha_i \in \mathbb{C}$, and $\Psi(x) = \psi_0 + \psi_1 x$, $\psi_j \in \mathbb{C}^{(m,m)}$.

The first aim of this section is to obtain explicit expressions for the elements associated with a zero class functional u in terms of the coefficients $\alpha_i \in \mathbb{C}$, $\psi_j \in \mathbb{C}^{(m,m)}$. This will lead to a characterization of the polynomials $\alpha \in \mathbb{P}_2 \setminus \{0\}$, $\Psi \in \mathbb{P}_1^{(m)}$ which can appear in the Pearson-type equation of a zero class functional. As a first restriction for α, Ψ, notice that Corollary 3.9 implies that $\psi_1 + n\alpha_2 I$ must be non-singular for $n \geq 0$.

Remember that (P_n) denotes the sequence of monic MOP related to u, $P_n(x) = x^n I + \pi_n x^{n-1} + \cdots$ and $E_n = \langle x^n P_n, u\rangle$. As we have shown in the proof of Corollary 3.17, the coefficients of the recurrence $xP_n = P_{n+1} + \beta_n P_n + \gamma_n P_{n-1}$ and the coefficients of the relation $P_n = \frac{1}{n+1}P'_{n+1} + a_n P'_n + b_n P'_{n-1}$ can be obtained from π_n and E_n. So, we will just calculate π_n and E_n in terms of α and Ψ.

From the Pearson-type equation for the functional u we obtain the relation (4) among the moments, that can be written in the following way

$$n\mu_{n-1}\alpha_0 + \mu_n N_n + \mu_{n+1} M_n = 0, \quad n \geq 0, \tag{15}$$

where $N_n = \psi_0 + n\alpha_1 I$, $M_n = \psi_1 + n\alpha_2 I$. Taking $n = 0$ and $n = 1$ in (15) we obtain

$$\mu_1 = -\mu_0\psi_0\psi_1^{-1}, \quad \mu_2 = \mu_0(\psi_0\psi_1^{-1}\psi_0 + \alpha_1\psi_0\psi_1^{-1} - \alpha_0)M_1^{-1}. \tag{16}$$

Let us denote $\tilde{u} = u\alpha I$ and $(\tilde{\mu}_n)_{n\geq 0}$ its corresponding moment sequence. We know that

$$\tilde{\mu}_n = \alpha_0\mu_n + \alpha_1\mu_{n+1} + \alpha_2\mu_{n+2}, \quad n \geq 0.$$

This equality for $n = 0$, together with (16), gives $\tilde{\mu}_0 = \mu_0\alpha(-\psi_0\psi_1^{-1})\psi_1 M_1^{-1}$. Besides, a direct calculation shows that $\pi_1 = -\mu_1\mu_0^{-1}$. So,

$$\pi_1 = E_0\psi_0\psi_1^{-1}E_0^{-1}, \qquad \tilde{\pi}_n = \tfrac{n}{n+1}\pi_{n+1},$$
$$\tilde{E}_0 = E_0\alpha(-\psi_0\psi_1^{-1})\psi_1 M_1^{-1}, \qquad \tilde{E}_n = -\tfrac{1}{n+1}E_{n+1}M_n,$$

where $\frac{1}{n+1}P'_{n+1}(x) = x^n + \tilde{\pi}_n x^{n+1} + \cdots$ and $\tilde{E}_n = \frac{1}{n+1}\langle x^n P'_{n+1}, \tilde{u}\rangle$.

Since u is a quasi-definite $\mathcal{P}_{2,1}$-functional, the same thing happens to $\tilde{u}$. Indeed, $\tilde{u}$ is also zero class because $D(\tilde{u}\alpha I) = u\tilde{\Psi}$, $\tilde{\Psi} = \Psi + \alpha' I$. Notice that $\tilde{\Psi}(x) = \tilde{\psi}_0 + \tilde{\psi}_1 x$, where $\tilde{\psi}_1 = M_2$ and $\tilde{\psi}_0 = N_1$.

The above results show that we can define a sequence $(u^{(j)})_{j\geq 0}$ of zero class functionals by $u^{(j)} = u\alpha^j$, and these functionals satisfy the Pearson-type equation

$$D(u^{(j)}\alpha) = u^{(j)}\Psi^{(j)}, \qquad \Psi^{(j)} = \Psi + j\alpha'.$$

Notice that $\psi_0^{(j)} = N_j$, $N_k^{(j)} = N_{k+j}$, $\psi_1^{(j)} = M_{2j}$, $M_k^{(j)} = M_{k+2j}$, where we denote with the superscript (j) the elements associated with the functional $u^{(j)}$. Therefore,

$$\begin{aligned} \pi_1^{(j)} &= E_0^{(j)} N_j M_{2j}^{-1} (E_0^{(j)})^{-1}, & \pi_k^{(j+1)} &= \tfrac{k}{k+1}\pi_{k+1}^{(j)}, \\ E_0^{(j+1)} &= E_0^{(j)}\alpha(-N_j M_{2j}^{(-1)}) M_{2j} M_{2j+1}^{-1}, & E_k^{(j+1)} &= -\tfrac{1}{k+1} E_{k+1}^{(j)} M_{k+2j}. \end{aligned}$$

After an inductive process,

$$\begin{aligned} \pi_n &= \pi_n^{(0)} = n\pi_1^{(n-1)} = nE_0^{(n-1)} N_{n-1} M_{2n-2}^{-1} (E_0^{(n-1)})^{-1}, \\ E_n &= E_n^{(0)} = (-1)^n n! E_0^{(n)} M_{2n-2}^{-1} \cdots M_{n-1}^{-1} = (-1)^n n! E_0^{(n)} M_{2n-1} V_{n-1}^{-1}, \end{aligned}$$

where $V_n = M_n \cdots M_{2n+1}$. Also,

$$E_0^{(n)} = E_0\alpha(-N_0 M_0^{-1}) M_0 M_1^{-1} \cdots \alpha(-N_{n-1} M_{2n-2}^{-1}) M_{2n-2} M_{2n-1}^{-1},$$

and, so,

$$E_n = (-1)^n n! E_0 \alpha(-N_0 M_0^{-1}) M_0 M_1^{-1} \cdots \alpha(-N_{n-1} M_{2n-2}^{-1}) M_{2n-2} V_{n-1}^{-1}. \quad (17)$$

If we define $\Pi_n = E_n^{-1}\pi_n E_n$, then

$$\begin{cases} \Pi_n = nV_{n-1} M_{2n-2}^{-1} N_{n-1} V_{n-1}^{-1}, \\ E_n^{-1} E_{n+1} = -(n+1) V_{n-1} M_{2n-1}^{-1} \alpha(-N_n M_{2n}^{-1}) M_{2n} V_n^{-1}. \end{cases} \quad (18)$$

The above expressions give π_n and E_n in terms of α and Ψ for a zero class functional u. When u satisfies the Pearson-type equation but it is not quasi-definite, the expressions for π_k and E_k are valid for the finite segment $(P_k)_{k=0}^n$ of MOP with respect to u, whenever $\Delta_0, \ldots, \Delta_n$ and $M_0, \ldots, M_{2n-1}$ are non-singular. This is because, then, the previous arguments remain valid for $(u^{(j)})_{j=0}^n$ and $(P_k^{(j)})_{k=0}^{n-j}$, as follows from Corollary 3.2 and Theorem 3.8.

Moreover, if M_{2n}, M_{2n+1} are non-singular too, the formulas are also valid for the coefficients π_{n+1}, E_{n+1} of the extra polynomial P_{n+1} orthogonal to $\mathbb{P}_n^{(m)}$, given by Proposition 2.3.

With the achieved results we can get a characterization of the polynomials α, Ψ related to the zero class.

Theorem 4.1. *The Pearson-type equation $D(u\alpha I) = u\Psi$, $\alpha \in \mathbb{P}_2 \setminus \{0\}$, $\Psi \in \mathbb{P}_1^{(m)}$, has a quasi-definite solution u if and only if M_n and $\alpha(-N_n M_{2n}^{-1})$ are non-singular for $n \geq 0$, where $N_n = \psi_0 + n\alpha_1 I$, $M_n = \psi_1 + n\alpha_2 I$. Under these conditions, the solution of the Pearson-type equation is unique up to left matrix factors, and the quasi-definite solutions correspond to the non-singular choices of μ_0.*

Proof. If $D(u\alpha I) = u\Psi$ has a quasi-definite solution, the corresponding matrices E_n are non-singular for $n \geq 0$. Then, M_n and $\alpha(-N_n M_{2n}^{-1})$ are non-singular for $n \geq 0$, as can be seen from (17).

For the converse, from Remark 3.11, if M_n is non-singular for $n \geq 0$, the solutions of the Pearson-type equation are determined by the choice of μ_0. Moreover, if, besides, $\alpha\left(-N_n M_{2n}^{-1}\right)$ is non-singular for $n \geq 0$, the solution u is quasi-definite when μ_0 is non-singular. Indeed, proceeding by induction we can prove that there exist MOP with respect to u of any degree:
· There exists $P_0 = I$, with $E_0 = \mu_0$ non-singular.
· Suppose that there exists a finite segment $(P_k)_{k=0}^n$ of monic MOP with respect to u. By Proposition 2.3, there is a monic matrix polynomial P_{n+1} with $\deg P_{n+1} = n+1$, which is orthogonal to $\mathbb{P}_n^{(m)}$. Since M_k is non-singular for $k \geq 0$, the expression of $E_{n+1} = \langle x^{n+1} P_{n+1}, u\rangle$ is given by (17). Then, the non-singularity of $\alpha\left(-N_k M_{2k}^{-1}\right)$ for $k \geq 0$ implies that E_{n+1} is non-singular and, hence, $(P_k)_{k=0}^{n+1}$ is also a finite segment of MOP with respect to u. □

Remark 4.2. From (17), we see that the non-singularity of M_k for $k \leq 2n-1$ and $\alpha(-N_j M_{2j}^{-1})$ for $j \leq n-1$, is equivalent to the existence of a finite segment $(P_k)_{k=0}^n$ of MOP with respect to any solution u of $D(u\alpha I) = u\Psi$ with μ_0 non-singular.

As in the classical scalar case, every matrix functional in the zero class belongs, up to a change of variable, to one of the following types:

- $\alpha(x) = 1$ Hermite-type polynomials.
- $\alpha(x) = x$ Laguerre-type polynomials.

- $\alpha(x) = 1 - x^2$ Jacobi-type polynomials.
- $\alpha(x) = x^2$ Bessel-type polynomials.

The characterization given by Theorem 4.1 can be particularized for any of the above canonical types. For the Hermite-type polynomials, the existence of a sequence of MOP is equivalent to the non-singularity of ψ_1. In the Laguerre case, ψ_1 and $\psi_0 + nI$ must be non-singular for $n \geq 0$. Jacobi-type polynomials exist if and only if $\psi_1 - nI$ and $\psi_1 \pm \psi_0 - 2nI$ are non-singular for $n \geq 0$, and, finally, the non-singularity of ψ_0 and $\psi_1 + nI$ for $n \geq 0$ characterizes the existence of the corresponding Bessel-type polynomials. Notice that the conditions for the existence of Hermite, Laguerre, Jacobi and Bessel-type MOP are a natural generalization of the conditions in the scalar case.

The non-singularity of the matrices M_n appeared previously in [13], as a condition for the Hermite, Laguerre and Jacobi-type polynomials to ensure that they are given by a Rodrigues formula. Our analysis proves that it is not necessary to impose this condition since it is automatically satisfied by any zero class functional.

Theorem 4.1 has also important practical consequences for the study of MOP. When a matrix functional is given by a positive definite weight matrix on $\mathbb{R}$, the corresponding MOP always exist. However, to decide if an arbitrary matrix of measures on $\mathbb{R}$ defines a quasi-definite functional can be a hard problem, even in the hermitian case. Theorem 4.1 solves this problem for any matrix functional satisfying a Pearson-type equation like (14). Moreover, Remark 4.2 gives a generalization that measures the length of the maximal finite segments of MOP associated with the functional when it is not quasi-definite. Some applications of this rule can be seen in Example 5. The importance of the above result for the zero class will be clear later, since we will see that the only non-trivial matrix functionals in this class are not positive definite.

4.1 Differential equation

In this section we will prove that the MOP of the zero class satisfy a second order differential equation that generalize the known one in the scalar case. Notice that this is not ensured by Theorem 2.12 *(iii)*, since the right-hand side of the differo-differential equation given by this theorem could have more than one term, as follows from the comments in Remark 2.13. We will also obtain the structure relation of Theorem 2.12 *(ii)*.

In order to obtain the differential equation, starting from the study of the family $\mathcal{P}_{2,1}$, and keeping in mind Corollary 3.17, we can write for any sequence (P_n) of MOP in the zero class,

$$P'_{n\pm1} = \Sigma_n^{(\pm)} P_n + \Gamma_n^{(\pm)} P'_n, \tag{19}$$

$$\begin{cases} \Sigma_n^{(+)} = (n+1) E_n M_{2n-1}^{-1} M_{n-1} E_n^{-1}, \\ \Sigma_n^{(-)} = -n E_{n-1} M_{2n-1}^{-1} M_{n-2} E_n^{-1}, \\ \Gamma_n^{(+)} = (n+1) E_n M_{2n-1}^{-1} (\alpha_2 E_n^{-1} x - \frac{1}{n} M_{2n-2} E_n^{-1} \pi_n + \frac{1}{n+1} M_{2n-1} E_n^{-1} \pi_{n+1}), \\ \Gamma_n^{(-)} = E_{n-1} M_{2n-1}^{-1} M_{n-2} E_n^{-1} (x + \frac{1}{n}\pi_n). \end{cases}$$

On the other hand, Theorem 2.12 *(ii)* and Remark 2.13 provide the structure relation

$$\alpha P'_n = n\alpha_2 P_{n+1} + \eta_n P_n + \theta_n P_{n-1}, \quad \eta_n, \theta_n \in \mathbb{C}^{(m,m)}. \tag{20}$$

Taking derivatives in the structure relation we obtain

$$\alpha P''_n + \alpha' P'_n = n\alpha_2 P'_{n+1} + \eta_n P'_n + \theta_n P'_{n-1}$$

and, using (19), we get

$$\alpha P''_n + (\alpha' I - \Gamma_n) P'_n - \Sigma_n P_n = 0, \tag{21}$$

$$\begin{cases} \Gamma_n = n\alpha_2 \Gamma_n^{(+)} + \theta_n \Gamma_n^{(-)} + \eta_n, \\ \Sigma_n = n\alpha_2 \Sigma_n^{(+)} + \theta_n \Sigma_n^{(-)}, \end{cases}$$

which is the differential equation for P_n.

We can calculate the coefficients of the above differential equation. First of all, notice that the coefficients η_n, θ_n of the structure relation can be expressed in terms of π_n and E_n. A direct computation from the structure relation (22) gives

$$\eta_n = n\alpha_1 + [(n-1)\pi_n - n\pi_{n+1}]\,\alpha_2, \qquad \theta_n = -E_n M_{n-1} E_{n-1}^{-1}.$$

Therefore, using (19), (21) and the above expressions, we find

$$\Sigma_n = n E_n M_{2n-1}^{-1} M_{n-1} [(n+1)\alpha_2 + M_{n-2}] E_n^{-1} = n E_n M_{n-1} E_n^{-1}.$$

In the same way, writing $\Gamma_n(x) = \Gamma_n^{(1)} x + \Gamma_n^{(0)}$, $\Gamma_n^{(i)} \in \mathbb{C}^{(m,m)}$, we get

$$\Gamma_n^{(1)} = E_n M_{2n-1}^{-1} \left[n(n+1)\alpha_2^2 - M_{n-1} M_{n-2} \right] E_n^{-1} = -E_n M_{-2} E_n^{-1},$$

$$\Gamma_n^{(0)} = n\alpha_1 - \frac{1}{n} E_n M_{2n-1}^{-1} \left[n(n+1)\alpha_2 M_{2n-2} + \right.$$
$$\left. + M_{n-1}M_{n-2} - n(n-1)\alpha_2 M_{2n-1} \right] E_n^{-1}\pi_n =$$
$$= n\alpha_1 - \frac{1}{n} E_n M_{2n-2} E_n^{-1}\pi_n = n\alpha_1 - \frac{1}{n} E_n M_{2n-2} \Pi_n E_n^{-1},$$

where Π_n is given in (18). From (18) and the above result we finally obtain

$$\alpha'(x) I - \Gamma_n(x) = E_n \psi_1 E_n^{-1} x + E_n V_{n-1} \psi_0 V_{n-1}^{-1} E_n^{-1}.$$

Summarizing, we can enunciate the following result.

Theorem 4.3. *Let u be a zero class functional with Pearson-type equation $D(u\alpha) = u\Psi$, $\alpha \in \mathbb{P}_2 \setminus \{0\}$, $\Psi \in \mathbb{P}_1^{(m)}$.*

(i) If (P_n) is the unique sequence of monic MOP with respect to u,

$$\alpha P_n'' + E_n V_{n-1} \Psi V_{n-1}^{-1} E_n^{-1} P_n' - n E_n M_{n-1} E_n^{-1} P_n = 0,$$

where $M_n = \psi_1 + n\alpha_2 I$ and $V_n = M_n M_{n+1} \cdots M_{2n+1}$.

(ii) If (Q_n) is the unique sequence of MOP whit respect to u such that Q_n has a leading coefficient $\kappa_n = (E_n V_{n-1})^{-1}$,

$$\alpha Q_n'' + \Psi Q_n' - n M_{n-1} Q_n = 0.$$

The differential equation satisfied by the MOP of the zero class characterizes such MOP, as the next result shows.

Theorem 4.4. *Let u be a zero class functional with Pearson-type equation $D(u\alpha I) = u\Psi$, $\alpha \in \mathbb{P}_2 \setminus \{0\}$, $\Psi \in \mathbb{P}_1^{(m)}$. Then, the differential equation*

$$\alpha y'' + \Psi y' - n M_{n-1} y = 0$$

has a unique (up to right matrix factors) matrix polynomic solution $y \in \mathbb{P}^{(m)}$. This solution is the only n-th MOP Q_n with respect to u which has a leading coefficient $\kappa_n = (E_n V_{n-1})^{-1}$.

Proof. Trying $y = \sum_{k \geq 0} c_k x^k$ as a solution of the differential equation, we obtain the recurrence for the coefficients

$$(n-k) M_{k+n-1} c_k = (k+1) \left[N_k c_{k+1} + (k+2)\alpha_0 c_{k+2} \right].$$

Since M_n is non-singular for $n \geq 0$, for every $k \neq n$, $c_{k+1} = c_{k+2} = 0$ implies $c_k = 0$. Hence, any non-trivial polynomic solution must have degree n, and such a solution is determined by c_n.

If $c_k = 0$ for $k > n$ and $c_n = \kappa_n$, there exists a unique solution that must be Q_n. If, on the contrary, $c_k = 0$ for $k > n$ but c_n is arbitrary, the solution is $Q_n L_n$, where $L_n = \kappa_n^{-1} c_n$. □

4.2 The hermitian case

Among all the zero class functionals, the hermitian ones have remarkable features that deserve to be emphasized. Maybe one of the most important has to do with the diagonalizatibility.

The main purpose of this section is to prove a conjecture of Durán and Grünbaum (see [13]): any positive definite zero class functional is diagonalizable by congruence. In fact, we will prove a more general result, since we will get the unitary diagonalizability and, at the same time, under much weaker conditions for the matrix functional. The key result to prove the referred conjecture is the following one.

Proposition 4.5. *Let $u \in \mathbb{P}^{(m)\prime}$ be a solution of $D\,(u\alpha I) = u\Psi$, $\alpha \in \mathbb{P}_2\backslash\{0\}$, $\Psi \in \mathbb{P}_1^{(m)}$. If $\mu_{n-2}, \ldots, \mu_{n+2}$ are hermitian,*

$$\psi_0^*\mu_{n+1}\psi_1 - \psi_1^*\mu_{n+1}\psi_0 = i2n(n+1)(A_0\mu_{n-1} + A_1\mu_n + A_2\mu_{n+1}),$$

with $A_0 = \Im(\bar{\alpha}_0\alpha_1),\ A_1 = 2\Im(\bar{\alpha}_0\alpha_2),\ A_2 = \Im(\bar{\alpha}_1\alpha_2)$.

Proof. From the hypothesis,

$$\langle \Psi^* x^n, u\Psi\rangle = \langle \Psi^* x^n, u\Psi\rangle^*.$$

Let us calculate $\langle \Psi^* x^n, u\Psi\rangle$.

$$\begin{aligned}
\langle \Psi^* x^n, u\Psi\rangle &= \langle \Psi^* x^n, D(u\alpha)\rangle = -n\langle \Psi^* x^{n-1}, u\alpha\rangle - \psi_1^*\langle x^n, u\alpha\rangle = \\
&= -n\langle \bar{\alpha} x^{n-1}, u\Psi\rangle^* - (\langle \bar{\alpha} x^{n-1}, u\Psi\rangle - \langle \bar{\alpha} x^{n-1}, u\rangle\psi_0)^* = \\
&= -(n+1)\langle \bar{\alpha} x^{n-1}, D(u\alpha)\rangle^* + \psi_0^*\langle x^{n-1}, u\alpha\rangle = \\
&= -(n+1)\langle \bar{\alpha} x^{n-1}, D(u\alpha)\rangle^* - \frac{1}{n}\psi_0^*\langle x^n, u\Psi\rangle.
\end{aligned}$$

Using the above results we get

$$(n+1)(\langle \bar{\alpha} x^{n-1}, D(u\alpha)\rangle - \langle \bar{\alpha} x^{n-1}, D(u\alpha)\rangle^*) = \frac{1}{n}(\psi_0^*\mu_{n+1}\psi_1 - \psi_1^*\mu_{n+1}\psi_0),$$

which, together with the equality

$$\langle \bar{\alpha} x^{n-1}, D(u\alpha)\rangle = -(n-1)\langle |\alpha|^2 x^{n-2}, u\rangle - \langle \bar{\alpha}'\alpha x^{n-1}, u\rangle,$$

gives

$$\begin{aligned}
\psi_0^*\mu_{n+1}\psi_1 - \psi_1^*\mu_{n+1}\psi_0 &= n(n+1)\langle (\bar{\alpha}\alpha' - \bar{\alpha}'\alpha)x^{n-1}, u\rangle = \\
&= i2n(n+1)\left[\Im(\bar{\alpha}_0\alpha_1)\mu_{n-1} + 2\Im(\bar{\alpha}_0\alpha_2)\mu_n + \Im(\bar{\alpha}_1\alpha_2)\mu_{n+1}\right].
\end{aligned}$$

□

Using the standard notation $[A, B] = AB - BA$ for the commutator of two square matrices A, B, we get the following immediate consequence of Proposition 4.5.

Corollary 4.6. *Under the conditions of Proposition 4.5, if* $\mu_0 = I$ *and* μ_1 *is hermitian too,*

$$\psi_1^*[\mu_{n+1}, \mu_1]\psi_1 = i2n(n+1)(A_0\mu_{n-1} + A_1\mu_n + A_2\mu_{n+1}),$$

with the coefficients A_0, A_1, A_2 *as in Proposition 4.5.*

The commutativity of a set of hermitian matrices is equivalent to state that they are simultaneously unitarily diagonalizable. Therefore, Corollary 4.6 relates the possibility of diagonalizing simultaneously μ_n and μ_1, to the requirement for α to have real coefficients. The next theorem gives conditions which ensure that α must be a real polynomial.

Remember that, if $\mu_0 > 0$ for a matrix functional, we can normalize it by $\mu_0 = I$ without loosing any hermiticity property of the functional. So, in what follows, we will use freely this normalization when it is possible.

Theorem 4.7. *Let* $u \in \mathbb{P}^{(m)\prime}$ *be a solution of* $D(u\alpha I) = u\Psi$, $\alpha \in \mathbb{P}_2 \setminus \{0\}$, $\Psi \in \mathbb{P}_1^{(m)}$. *If* $\mu_n = \mu_n^*$ *for* $n \leq 5$, *then* α *is a real polynomial (up non-trivial factors) under any of the followings conditions:*

(i) $[\mu_2, \mu_1] = 0$, $\Delta_0 > 0$ *and* $\Delta_1, \dots, \Delta_5$ *non-singular.*

(ii) $\Delta_2 > 0$.

Proof. Without loss of generality, we can suppose $\mu_0 = I$. Let A_0, A_1, A_2 be the coefficients given in Proposition 4.5.

(i) $[E_1, \mu_1] = 0$ since $E_1 = \mu_2 - \mu_1^2$. Then, from (17) for $n = 1$, we obtain $[\psi_1, \mu_1] = 0$, which implies $[\psi_0, \mu_1] = 0$ because $\psi_0 = -\mu_1\psi_1$. Using (15) and the fact that M_n is non-singular for $n \leq 3$, due to Theorem 3.8, we get $[\mu_n, \mu_1] = 0$ for $n \leq 4$. Then, from Corollary 4.6,

$$\Delta_2 \begin{pmatrix} A_0 \\ A_1 \\ A_2 \end{pmatrix} = 0,$$

which implies $A_i = 0$, $\forall i$.

(ii) Corollary 4.6 for $n = 1, 2, 3$ gives

$$\Delta_2 \begin{pmatrix} A_0 \\ A_1 \\ A_2 \end{pmatrix} = \frac{1}{24i} \begin{pmatrix} 6\psi_1^*[\mu_2, \mu_1]\psi_1 \\ 2\psi_1^*[\mu_3, \mu_1]\psi_1 \\ \psi_1^*[\mu_4, \mu_1]\psi_1 \end{pmatrix}.$$

Therefore,

$$(A_0 \; A_1 \; A_2) \, \Delta_2 \begin{pmatrix} A_0 \\ A_1 \\ A_2 \end{pmatrix} = \frac{1}{24i} \psi_1^* \left(6A_0 \, [\mu_2, \mu_1] + 2A_1 \, [\mu_3, \mu_1] + A_2 \, [\mu_4, \mu_1] \right) \psi_1 .$$

Notice that, if $P(x) = (A_0 + A_1 x + A_2 x^2) I$,

$$\langle P, uP^* \rangle = (A_0 \; A_1 \; A_2) \, \Delta_2 \begin{pmatrix} A_0 \\ A_1 \\ A_2 \end{pmatrix} .$$

Let us suppose $P \neq 0$. Since $\Delta_2 > 0$, Proposition 2.5 implies that $\langle P, uP^* \rangle > 0$. From Lemma 3.3 we know that ψ_1 is non-singular, so, the matrix $(\psi_1^{-1})^* \langle P, uP^* \rangle \psi_1^{-1}$ must be positive definite too. On the other hand, $\operatorname{tr} [\mu_n, \mu_1] = 0$ and, thus, $\operatorname{tr} ((\psi_1^{-1})^* \langle P, uP^* \rangle \psi_1^{-1}) = 0$. Hence, $\langle P, uP^* \rangle$ can not be positive definite. This means that $P = 0$ and $A_i = 0$, $\forall i$. □

Corollary 4.8. *For any positive definite zero class functional, the scalar polynomial of the Pearson-type equation is real up to non-trivial factors.*

The following result says that a zero class functional with a real scalar polynomial in the Pearson-type equation does not need to many conditions to be unitarily diagonalizable.

Theorem 4.9. *Let u be a zero class functional with $\mu_n = \mu_n^*$ for $n \leq 3$ and $\Delta_0 > 0$. Then, if the scalar polynomial α of the Pearson-type equation is real up to factors, u is unitarily diagonalizable.*

Under the above conditions, if μ_4, μ_5 are hermitian too, then, α is real up to factors if and only if u is unitarily diagonalizable.

Proof. Suppose without loss of generality that $\mu_0 = I$. If $A_i = 0$, $\forall i$, Corollary 4.6 for $n = 1$ gives $\psi_1^* \, [\mu_2, \mu_1] \, \psi_1 = 0$. Since ψ_1 is non-singular, $[\mu_2, \mu_1] = 0$, so, there exists $T \in \mathbb{C}^{(m,m)}$ unitary such that $T \mu_n T^*$ is diagonal for $n = 1, 2$. Then, $T E_1 T^*$ is diagonal because $E_1 = \mu_2 - \mu_1^2$. From (17) for $n = 1$ we find that $T \psi_1 T^*$ is diagonal too. Hence, $T \psi_0 T^*$ is also diagonal due to the identity $\psi_0 = -\mu_1 \psi_1$. Using (15) and the non-singularity of M_n for $n \geq 0$ one finds that $T \mu_n T^*$ is diagonal for $n \geq 0$.

The converse when μ_4, μ_5 are hermitian follows from Theorem 4.7 *(i)*. □

Joining Theorem 4.7 and 4.9 we achieve the following result that goes even further than the conjecture of Durán and Grünbaum.

Theorem 4.10. *Let u be a zero class functional with $\mu_n = \mu_n^*$ for $n \leq 5$. Then, u is unitarily diagonalizable under any of the followings conditions:*

(i) $\Delta_0 > 0$ *and* $[\mu_2, \mu_1] = 0$.

(ii) $\Delta_2 > 0$.

Notice that some of the conditions in Theorems 4.7, 4.9 and 4.10 can be weakened. For example, in Theorem 4.7 *(i)*, it is possible to substitute the condition $\Delta_1, \dots, \Delta_5$ non-singular by Δ_2 non-singular and $[\mu_3, \mu_1] = [\mu_4, \mu_1] = 0$.

Corollary 4.11. *(Durán-Grünbaum conjecture) Any positive definite zero class functional is unitarily diagonalizable.*

The above result does not mean that the hermitian zero class is trivial, since there exist non-diagonalizable zero class MOP with respect to hermitian functionals which are not positive definite (see Example 5). What is trivial is the positive definite subclass of the zero class (actually, a bigger subclass, according to Theorem 4.10). Hence, positive definite Hermite, Laguerre and Jacobi-type MOP are unitarily diagonalizable. Concerning the Bessel case we can say even something more: similarly to the scalar situation, positive definite Bessel-type MOP do not exist, as the following proposition asserts.

Proposition 4.12. *Any zero class functional whose Pearson-type equation has a scalar polynomial with a double root, is not positive definite.*

Proof. Assume that u is a positive definite zero class functional whose corresponding Pearson-type equation has a scalar polynomial $\alpha(x) = (x-a)^2$, $a \in \mathbb{C}$. From Corollary 4.8, $a \in \mathbb{R}$. Also, Corollary 4.11 implies that there exists $T \in \mathbb{C}^{(m,m)}$ unitary such that $T\mu_n\mathbb{T}^*$ is diagonal for $n \geq 0$. Therefore, TE_1T^* is also diagonal and, using (17), we get that $T\psi_1T^*$ and $T\psi_0T^*$ are diagonal too. So, if we define the change of variable $t(x) = x - a$, the diagonal hermitian matrix functional $\hat{u}_t = Tu_tT^*$ satisfies the Pearson-type equation $D(\hat{u}_t\, t^2 I) = \hat{u}_t T\Psi(t+a)T^*$. Hence, $\hat{u}_t = \hat{u}_t^{(1)} \oplus \cdots \oplus \hat{u}_t^{(m)}$, where $\hat{u}_t^{(i)}$ are scalar Bessel functionals. Since a scalar Bessel functional can not be positive, the functional u is not positive definite, in contradiction with the hypothesis. □

4.3 Examples

An example of non-diagonalizable hermitian zero class functional was presented in [5]. [13] generalizes this example and provides several non-trivial

families of hermitian matrix functionals that satisfy a Pearson-type equation like (14). In this section we will use these examples, including some non-hermitian generalizations, and we will prove that the corresponding zero class MOP do exist as an application of Theorem 4.1. Notice that [13] does not answer this question since the analysis of the non-positive definite weights $dM(x)$ given there was under the assumption that $\int_{\mathbb{R}} P(x)\,dM(x)P(x)$ is non-singular for any matrix polynomial P with non-singular leading coefficient, something that was not proved in the concrete examples.

The non-diagonalizability of the functionals given in the following examples is ensured because they have the structure $u = W(x)\,dx$, where

$$W = \begin{pmatrix} w_{11} & w_{12} \\ w_{21} & 0 \end{pmatrix}$$

with $\{w_{11}, w_{12}\}$ linearly independent and $\{w_{12}, w_{21}\}$ linearly dependent. These conditions imply that the functional u is not diagonalizable by congruence or, even, by equivalence.

Example 5. Let us consider a functional $u \in \mathbb{P}^{(2)'}$ given by $u = w(x)R(x)\,dx$, where w is a positive classical scalar weight with Pearson equation $(w\alpha)' = w\beta$ and

$$R(x) = \begin{pmatrix} c + \int \frac{q(x)}{\alpha(x)}\,dx & a \\ b & 0 \end{pmatrix}, \quad q \in \mathbb{P}_1 \setminus \{0\}, \quad a, b \in \mathbb{C} \setminus \{0\}, \quad c \in \mathbb{C}.$$

Notice that u is hermitian when $b = \bar{a}$, $c \in \mathbb{R}$ and q is a real polynomial.

This kind of functionals always satisfy the boundary conditions which ensure that $D(u\alpha I) = (u\alpha I)'$ (see Remark 2.9). In fact, writing them in the canonical representations, they have the form

$$e^{-x^2} \begin{pmatrix} c + c_1 x + c_2 x^2 & a \\ b & 0 \end{pmatrix} dx, \quad x \in \mathbb{R},$$
$$x^r e^{-x} \begin{pmatrix} c + c_1 x + c_2 \log(x) & a \\ b & 0 \end{pmatrix} dx, \quad x \in (0, \infty),$$
$$(1+x)^r (1-x)^s \begin{pmatrix} c + c_1 \log(1+x) + c_2 \log(1-x) & a \\ b & 0 \end{pmatrix} dx, \quad x \in (-1, 1),$$

in the Hermite, Laguerre and Jacobi case respectively. In the above expressions $c_1, c_2 \in \mathbb{C}$ do not vanish simultaneously and $r, s > -1$.

The functional u satisfies the Pearson-type equation

$$D(u\alpha I) = u\Psi, \quad \Psi = \begin{pmatrix} \beta & 0 \\ \frac{q}{a} & \beta \end{pmatrix}.$$

Therefore, if $q(x) = q_0 + q_1 x$ and $\beta(x) = \beta_0 + \beta_1 x$,

$$M_n = \begin{pmatrix} \beta_1 + n\alpha_2 & 0 \\ \frac{q_1}{a} & \beta_1 + n\alpha_2 \end{pmatrix},$$

$$\alpha(-N_n M_{2n}^{-1}) = \alpha\left(-\frac{\beta_0 + n\alpha_1}{\beta_1 + 2n\alpha_2}\right)\begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix}.$$

Notice that, due to Theorem 4.1, $\beta_1 + n\alpha_2$ and $\alpha(-\frac{\beta_0+n\alpha_1}{\beta_1+2n\alpha_2})$ must be different from zero for $n \geq 0$. Hence, M_n and $\alpha(-N_n M_{2n}^{-1})$ are non-singular for $n \geq 0$. Also, μ_0 is non-singular since

$$\mu_0 = \nu_0 \begin{pmatrix} * & a \\ b & 0 \end{pmatrix}, \quad \nu_0 = \int_{\mathbb{R}} w(x)\, dx.$$

So, according to Theorem 4.1, we conclude that the functional u defines a sequence of zero class MOP.

The above two-dimensional examples are only particular cases of the m-dimensional zero class functionals belonging to the equivalence classes defined by

$$\begin{array}{l} e^{Ax}e^{-Bx^2}dx, \quad x \in \mathbb{R}, \quad \Re(\lambda) > 0 \;\; \forall \lambda \in \operatorname{spec}(B), \\ x^A e^{-Bx} dx, \quad x \in (0,\infty), \quad \begin{cases} \Re(\lambda) > -1 \;\; \forall \lambda \in \operatorname{spec}(A), \\ \Re(\lambda) > 0 \;\; \forall \lambda \in \operatorname{spec}(B), \end{cases} \\ (1+x)^A (1-x)^B dx, \quad x \in (-1,1), \quad \Re(\lambda) > -1 \;\; \forall \lambda \in \operatorname{spec}(A), \operatorname{spec}(B), \end{array}$$

where $A, B \in \mathbb{C}^{(m,m)}$ commute and $\operatorname{spec}(A)$ means the spectrum of the matrix A. The restrictions for the spectra ensure the integrability for any matrix polynomial and, together with the commutativity of A and B, lead to a Pearson-type equation of Hermite, Laguerre and Jacobi-type respectively, according to Remark 2.9. The conditions for the spectra also ensure the existence of MOP whenever μ_0 is non-singular, as follows from Theorem 4.1. For some choices of A and B it is possible to get an equivalent hermitian functional. This is the case of the initial examples, as [13] points out.

These examples do not cover the zero class functionals of Bessel-type. Such examples can be found starting from a scalar Bessel weight. For instance, $w(x) = x^r e^{1/x}$, with $r = -1, 0, 1, 2, \dots$, is a Bessel weight on the unit circle $\mathbb{T} := \{x \in \mathbb{C} \mid |x| = 1\}$ with Pearson equation $(w\alpha)' = w\beta$, $\alpha(x) = x^2$, $\beta(x) = (r+2)x - 1$. The matrix function $W = wR$ satisfies the equation $(W\alpha)' = W\Psi$, where R and Ψ have the same meaning as previously. However,

$$W(x) = x^r e^{1/x} \begin{pmatrix} c + \frac{c_1}{x} + c_2 \log(x) & a \\ b & 0 \end{pmatrix}$$

is not analytic on $\mathbb{T}$ if $c_2 \neq 0$. If, for instance, we choose a logarithm with the discontinuity at the non-negative real axis, the matrix functional $u = W(x)\,dx$, $x \in \mathbb{T}$, verifies (see Remark 2.9)

$$D(u\alpha I) = (W\alpha)'(x)\,dx - i2\pi e c_2 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \delta(x-1)\,dx,$$

so, it satisfies the Pearson-type equation $D(u\alpha I) = u\Psi$ when $c_2 = 0$.

Similarly to the initial examples, this new one is equivalent to a particular two-dimensional case of the general m-dimensional zero class functionals with the form $x^r e^{B/x} dx$, $x \in \mathbb{T}$, where $r = -1, 0, 1, 2, \ldots$ and $B \in \mathbb{C}^{(m,m)}$ is non-singular. Analogously to the scalar case, these functionals satisfy a Pearson-type equation of Bessel-type since the restriction on r gives the analyticity on $\mathbb{T}$ for $x^r e^{B/x}$. As in the previous examples, the conditions for r and B ensure the existence of the corresponding MOP when μ_0 is non-singular, due to Theorem 4.1.

Concerning the restriction on r it is known that, for the Bessel scalar case, it can be weakened to $r \neq -2, -3, \ldots$ by introducing the alternative weight on $\mathbb{T}$

$$w_0(x) = \sum_{k=0}^{\infty} \frac{\Gamma(r+2)}{\Gamma(r+2+k)} \frac{1}{x^{k+1}}.$$

This weight satisfies the equation $(w_0\alpha)' = w_0\beta + r + 1$, $\alpha(x) = x^2$, $\beta(x) = (r+2)x - 1$. So, according to Remark 2.9, the scalar functional $u_0 = w_0(x)\,dx$, $x \in \mathbb{T}$, verifies the Pearson-type equation $D(u_0\alpha) = u_0\beta$.

Notice that $\frac{\Gamma(r+2)}{\Gamma(r+2+k)} = \frac{1}{(r+2)_k}$ where, in general, we denote

$$(A)_k = \begin{cases} I & \text{if } k = 0, \\ A(A+I)\cdots(A+(k-1)I) & \text{if } k \in \mathbb{N}, \end{cases}$$

for any square matrix A. If $A, B \in \mathbb{C}^{(m,m)}$ and $\operatorname{spec}(A) \cap \{0, -1, -2, \ldots\} = \emptyset$, we can consider the matrix function

$$W(x) = \sum_{k=0}^{\infty} (A)_k^{-1} B^k \frac{1}{x^{k+1}},$$

which is analytical on $\mathbb{C} \setminus \{0\}$. If, besides, A and B commute, then $(W\alpha)' = W\Psi + A - I$, $\alpha(x) = x^2$, $\Psi(x) = Ax - B$. Hence, the matrix functional $u = W(x)\,dx$, $x \in \mathbb{T}$, satisfies the Pearson-type equation $D(u\alpha I) = u\Psi$ analogously to the scalar case. Therefore, Theorem 4.1 states that there exist Bessel-type MOP associated with u when B and μ_0 are non-singular.

5 Other differential equations

Among the results proved by Durán in [10], we remark in this section one concerning the existence of differential equations for MOP with respect to hermitian functionals $u \in \mathbb{P}^{(m)'}$ satisfying a Pearson-type equation

$$D(u\Phi) = u\Psi, \quad \Phi \in \mathbb{P}_2^{(m)}, \quad \Psi \in \mathbb{P}_1^{(m)}.$$

The referred result states that such a Pearson-type equation, together with the hermiticity of $u\Phi$, is equivalent to state that the corresponding MOP (P_n) satisfy a second order differential equation

$$P_n''\Phi^* + P_n'\Psi^* + \Lambda_n P_n = 0, \tag{22}$$

with $\Lambda_n \in \mathbb{C}^{(m,m)}$ such that $\Lambda_n \langle P_n, P_n \rangle_u$ is hermitian (actually, the result is proved in [10] for matrix orthonormal polynomials with respect to positive definite matrix functionals, but the generalization to the quasi-definite hermitian case is immediate). If, as in the rest of paper, we suppose that the MOP are monic, the condition for Λ_n becomes $\Lambda_n E_n = E_n \Lambda_n^*$. Also, equaling the coefficients of the highest powers of x in (22) we get $\Lambda_n = -n(n-1)\psi_1^* - n\varphi_2^* = -nM_{n-1}^*$.

All the examples of $\mathcal{P}_{2,1}$-functionals $u \in \mathbb{P}^{(2)'}$ presented in Section 3 were hermitian and positive definite and, for all of them, we found a matrix polynomial $\Phi \in \mathcal{M}_{2,1}(u)$ with $\det \Phi \neq 0$ such that $u\Phi$ is also hermitian and positive definite (in Examples 2 and 4 such a matrix polynomial was denoted $\Phi^{(0)}$, we omit now the superscript for convenience). Therefore, the corresponding MOP (P_n) must satisfy a second order differential equation like (22).

For instance, in the case of the functional given in Example 2

$$u = e^{-x^2} \begin{pmatrix} 1 + |a|^2 x^2 & ax \\ \bar{a}x & 1 \end{pmatrix} dx, \quad x \in \mathbb{R}, \quad a \in \mathbb{C} \setminus \{0\}.$$

we find

$$P_n''(x) \begin{pmatrix} |a|^2 + 2 & -a|a|^2 x \\ 0 & 2 \end{pmatrix} + P_n'(x) \begin{pmatrix} -4x & 2a \\ 2\bar{a} & -2(|a|^2+2)x \end{pmatrix} + \\ + n \begin{pmatrix} 4 & 0 \\ 0 & 2(|a|^2+2) \end{pmatrix} P_n(x) = 0.$$

This functional was previously studied in [14], where it was proved that the corresponding MOP satisfy another second order differential equation linearly independent with respect to this one. The fact that, contrary to the

scalar case, the MOP can satisfy linearly independent second order differential equations was recently discovered (see [7, 16]).

As for the functional

$$u = x^r e^{-x} \begin{pmatrix} x + |a|^2 x^2 & ax \\ \bar{a}x & 1 \end{pmatrix} dx, \quad x \in (0,\infty), \quad a \in \mathbb{C} \setminus \{0\}, \quad r > -1,$$

given in Example 3, we get

$$\begin{aligned} P_n''(x) &\begin{pmatrix} (|a|^2+1)x & -a|a|^2x^2 \\ 0 & x \end{pmatrix} + \\ &+ P_n'(x) \begin{pmatrix} (r+2)(|a|^2+1) - x & -(r+2)a|a|^2 x \\ \bar{a} & r+1-(|a|^2+1)x \end{pmatrix} + \\ &+ n \begin{pmatrix} 1 & (r+1+n)a|a|^2 \\ 0 & |a|^2+1 \end{pmatrix} P_n(x) = 0. \end{aligned}$$

Finally, Example 4 deals with the functional

$$u = x^r e^{-x} \begin{pmatrix} x^2 + |a|^2 x^2 & ax \\ \bar{a}x & 1 \end{pmatrix} dx, \quad x \in (0,\infty), \quad a \in \mathbb{C} \setminus \{0\}, \quad r > -1,$$

whose MOP must satisfy the differential equation

$$\begin{aligned} &P_n''(x) \begin{pmatrix} (r+1)x & 0 \\ -\bar{a} & (r+|a|^2+2)x \end{pmatrix} + \\ &+ P_n'(x) \begin{pmatrix} (r+1)[(r+|a|^2+3) - x] & -(r+1)a(|a|^2+1)x \\ \bar{a} & (r+1)(r+2) - (r+|a|^2+2)x \end{pmatrix} + \\ &+ n \begin{pmatrix} r+1 & (r+1)a(|a|^2+1) \\ 0 & r+|a|^2+2 \end{pmatrix} P_n(x) = 0. \end{aligned}$$

Let us restrict our attention now to the zero class MOP, that is, those whose corresponding functional $u \in \mathbb{P}^{(m)'}$ satisfies a Pearson-type equation

$$D(u\alpha I) = u\Psi, \quad \alpha \in \mathbb{P}_2 \setminus \{0\}, \quad \Psi \in \mathbb{P}_1^{(m)}.$$

If u is hermitian, the hermiticity of $u\alpha I$ is equivalent to saying that α is a real polynomial. Hence, if u is hermitian and α is real, the MOP (P_n) with respect to u satisfy the second order differential equation

$$\alpha P_n'' + P_n' \Psi^* - n M_{n-1}^* P_n = 0.$$

This differential equation is similar, but not equal to the one given in Theorem 4.3. However, when $\mu_0 > 0$ this difference disappears since, then,

Theorem 4.9 implies that u is unitarily diagonalizable. That is, there exists $T \in \mathbb{C}^{(m,m)}$ unitary such that $\hat{u} = TuT^*$ is diagonal hermitian, so, the corresponding monic MOP $(\hat{P}_n)$ must be diagonal with real polynomials in the diagonal. Following similar arguments to those given in the proofs of the theorems in Section 4, we find that $\hat{\Psi} = T\Psi T^*$ is also diagonal. Moreover, $D(\hat{u}\alpha I) = u\hat{\Psi}$, hence, $\hat{\Psi}$ is real. Therefore, both differential equations are the same for $(\hat{P}_n)$ and, thus, also for (P_n) since $\hat{P}_n = TP_nT^*$.

Returning to the family $\mathcal{P}_{2,1}$, the two-dimensional examples that we have found suggest that, for a big subclass of hermitian $\mathcal{P}_{2,1}$-functionals, the related MOP satisfy a second order differential equation like (22). Equivalently, it seems that for many hermitian $\mathcal{P}_{2,1}$-functionals $u \in \mathbb{P}^{(m)'}$ it is possible to find a generator Φ of the module $\mathcal{M}_{2,1}(u)$ such that $u\Phi$ is hermitian too. In particular, the referred examples seem to indicate that if u is positive definite, then $u\Phi$ is also positive definite for some generator Φ of $\mathcal{M}_{2,1}(u)$. The characterization of the subclasses of hermitian $\mathcal{P}_{2,1}$-functionals which are invariant under the operation $u \to u\Phi$ (for some choice of the generator Φ of $\mathcal{M}_{2,1}(u)$) remains as an open problem. This is an important question, not only for the study of differential equations for MOP, but also for the development of a general and systematic method to obtain modified Rodrigues' formulas for $\mathcal{P}_{2,1}$-functionals (see [14] for some examples of this kind of Rodrigues' formulas), as it will be shown in a future paper.

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